Uncertainty, Contracting, and Beliefs in Organizations

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David L. Dicks
Baylor University

Paolo Fulghieri
University of North Carolina, CEPR and ECGI

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Abstract

A multidivisional firm has headquarters exposed to moral hazard by division managers under uncertainty. Uncertainty creates endogenous disagreement aggravating moral hazard; by hedging uncertainty, headquarters design incentive contracts that reduce disagreement, lower incentive provision costs and promote effort. Because hedging uncertainty can conflict with hedging risk, optimal contracts differ from standard principal-agent models. Our model helps explain the prevalence of equity-based incentive contracts and the rarity of relative performance contracts, especially in firms facing greater uncertainty. Finally, we show the aggregation and linearity properties of Holmström and Milgrom (1987) hold in a dynamic model under IID ambiguity of Chen and Epstein (2002).

Keywords: Contracting, Organizations, Hierarchy, Uncertainty Aversion, Ambiguty Aversion

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David L. Dicks*
Assistant Professor
Baylor University, Hankamer School of Business
Forster Business and Innovation
Waco, Texas 76706, United States
phone: +1 (254) 710-4771
e-mail: David_Dicks@baylor.edu

Paolo Fulghieri
Macon G. Patton Distinguished Professor of Finance
University of North Carolina Kenan
300 Kenan Center Drive
Chapel Hill, NC 27599-3490, United States
phone: +1 919-962-3202
e-mail: Paolo_Fulghieri@unc.edu

*Corresponding Author

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David L. Dicks
Baylor University

Paolo Fulghieri
University of North Carolina
CEPR and ECGI

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Abstract

A multidivisional firm has headquarters exposed to moral hazard by division managers under uncertainty. Uncertainty creates endogenous disagreement aggravating moral hazard; by hedging uncertainty, headquarters design incentive contracts that reduce disagreement, lower incentive provision costs and promote effort. Because hedging uncertainty can conflict with hedging risk, optimal contracts differ from standard principal-agent models. Our model helps explain the prevalence of equity-based incentive contracts and the rarity of relative performance contracts, especially in firms facing greater uncertainty. Finally, we show the aggregation and linearity properties of Holmström and Milgrom (1987) hold in a dynamic model under IID ambiguity of Chen and Epstein (2002).

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The provision of incentives in organizations is essential for economic efficiency. A key question is to determine appropriate performance measures for incentive pay. Managerial contracts often are a combination of base-pay, based on narrowly defined division-specific performance measures ("pay-for-performance"), plus a component linked to overall firm profitability (i.e. bonuses, equity-based pay, and other "aggregate" performance measures).[1] This distinction is particularly important for lower-level managers. The case for equity-based incentives for top managers is rather strong as they are responsible for the performance of the overall firm. Absent inter-dependencies across divisions, the use of equity-based pay for division managers and rank-and-file employees is more puzzling. For lower-level employees, equity-based compensation reduces responsiveness of pay to actions, weakening incentives at the cost of increasing their overall risk exposure. In addition, when cash-flows are positively correlated across divisions, to reduce harmful risk bearing incentive contracts should display a relative-performance component, a feature more rarely observed in practice.

We study the impact of uncertainty (or "ambiguity") aversion on the design of incentive contracts in organizations.[2] Our key feature is to acknowledge that most corporate decisions are taken without full knowledge of the probability distributions involved, a situation characterized as uncertainty (Knight, 1921). We consider a multi-division firm with headquarters, HQ, and (two) division managers. Division cash-flows depend on unobservable effort exerted by division managers, and can be (positively or negatively) correlated. Division managers (and HQ) are uncertain on division productivity, affecting their incentives to exert effort. To isolate the effect of uncertainty on incentive pay, we rule out synergies or other inter-dependencies across divisions (as in Holmström, 1982).

Traditional principal-agent theory (Holmström, 1979, 1982*)* suggests that, to limit risk exposure, incentive contracts should depend only on performance measures that are informative on actions (the "informativeness principle," Holmström, 2017).[3] An implication is that incentive contracts should hedge division managers' risk by giving a negative (positive) exposure (only) to variables positively (negatively) correlated to division cash-flow's residual risk. In our setting, HQ can use

[1]The use of aggregate performance measures, such as bonuses, is documented in the accounting literature (Bushman et al., 1995, Bouwens and Van Lent, 2007, and Labro and Omartian, 2022). See Frydman and Jenter (2010), Oyer and Schaefer (2011), Murphy (2013), and Edmans et al. (2017) for extensive surveys.

[2]The importance of ambiguity aversion in affecting individual decision making has been shown in both experimental and empirical studies (e.g., Bossaerts et al., 2010, Hong et al., 2018, Anderson et al., 2009, Ju and Miao, 2012, Jeong et al, 2015, Epstein and Schneider, 2008, and Machina and Siniscalchi, 2014).

[3]Responsiveness of CEO pay to risk factors not informative on their actions ("pay-for-luck") has been documented by several studies (e.g., Bertrand and Mullainathan, 2001; Choi, Gipper and Shi, 2020).

cross-pay to (partially) hedge division-manager risk exposure. Risk hedging can be obtained by offering contracts with a relative-performance component for a positive correlation of division cash flows, or an equity-based component for a negative correlation.

These predictions change substantially in the presence of uncertainty aversion. We model uncertainty aversion by adopting the multiple prior approach of Gilboa and Schmeidler (1989). In this setting, uncertainty-averse agents do not have a single prior but, rather, are endowed with a set of admissible priors (the "core belief set") and assess random variables by selecting, from that set, the measure that minimizes their expected utility. We model the core-beliefs set on the basis of the relative entropy criterion of Hansen and Sargent (2001) and (2008). Intuitively, under the relative entropy criterion uncertainty-averse agents consider as admissible only probability measures that are not "too unlikely" to be the true distribution given a certain reference probability.

The presence of uncertainty creates endogenous disagreement between HQ and division managers, and has two adverse effects.[4] First, traditional incentive contracts, by loading primarily on division cash-flows, lead ambiguity-averse managers to hold more conservative estimates (beliefs) than HQ on the productivity of their own division, with a negative impact on their effort. Similar to Miao and Rivera (2016), we interpret "beliefs" broadly, as the probability measure that agents adopt to assess random variables and consequences of actions. More conservative beliefs are due to division managers' greater exposure to uncertainty on their own division than HQ, who instead have exposure to the overall firm. The implication is that HQ must increase pay-for-performance sensitivity to elicit any desired level of effort. Second, disagreement with HQ leads division managers to value compensation contracts at discount with respect to the value attributed by the (more confident) HQ, which makes it more difficult to meet their participation constraint, increasing the cost of incentive provision.

We argue that HQ can reduce the negative impact of disagreement by managing individual exposure to uncertainty through contracts, with beneficial effects on incentives. The role of contracts in managing agents' beliefs is novel in the theory of contract design. It is a direct consequence of the property that beliefs held by uncertainty-averse agents are determined endogenously and depend

[4] In a single-agent principal-agent model with uncertainty-averse principal and uncertainty neutral agent, Miao and Rivera (2016) show that uncertainty aversion causes disagreement between principal and agent, affecting incentive contract design. In our model, we consider a principal-agent problem with multiple agents, where both principal and agents are uncertainty averse.

on their exposure to the sources of uncertainty. We interpret "beliefs" broadly, as the probability measure that agents adopt to assess random variables and consequences of actions. Differential exposure to uncertainty may be due to the position in the organization (hierarchical exposure) or to the contractual relationships that bind agents (contractual exposure). Hierarchical and contractual exposure concur together to determine the prevailing structure of beliefs in an organization. By design of incentive contracts, HQ can affect agents beliefs with a positive impact on incentives. An implication is that equity-based incentive contracts can be used to realign internal beliefs, which generates consensus by promoting a "shared view" in the organization.[5] The presence of a shared view can reinforce the beneficial effect of equity in fostering internal cooperation.

The key economic driver in our paper is that uncertainty-averse division managers hold (weakly) more favorable expectations on their own division, and thus are more confident, when incentive pay depends on the performance of both divisions, that is, with cross-pay. This positive effect on beliefs is a consequence of the benefits of uncertainty hedging that stem from the "uncertainty aversion" axiom of Gilboa and Schmeidler (1989). Intuitively, pay-for-performance in incentive pay makes uncertainty-averse division managers concerned that the productivity of their own division is extremely low, depressing their effort incentives. The presence of cross-pay hedges the uncertainty faced by division managers. Consider, for example, an incentive contract with an equity-pay component. The presence of equity-based pay makes division managers exposed to uncertainty from *both* divisions and will regard the possibility that *both* divisions are characterized by extremely low productivity sufficiently unlikely to be ruled out by the relative entropy criterion. The effect is to make division managers hold more favorable beliefs on their own division (in fact on both divisions), improving their effort incentives. A similar effect of ruling out extreme beliefs can be obtained with a relative-performance component in compensation, where division managers effectively have a negative (or a "short") exposure in the other division (with the difference that now it will lead division managers to hold very unfavorable beliefs on the other division, due to their short position).

[5]The role of equity-based compensation to promote consensus in organizations is examined in Organization Behavior literature, such as Klein (1987), Pearsall, Christian, and Ellis (2010), and Blasi, Freeman, and Kruse (2016), among others. The importance of promoting a shared view is discussed in Zohar and Hofmann (2012). Advantages and disadvantages of disagreement in organizations has been studied in several papers: Dessein and Santos (2006); Landier, Sraer, and Thesmar (2009); Bolton, Brunnermeier, and Veldkamp (2013); and Van den Steen (2005) and (2010).

We show that optimal contracts depend on the level of uncertainty faced by division managers and HQ. For expositional simplicity, we restrict our analysis to linear incentive contracts in a static problem.[6] In the simpler case where HQ are uncertainty neutral, optimal contracts depend on the extent of division managers' exposure to uncertainty and on the sign of the correlation between division cash-flows. When division managers face low uncertainty, incentive contracts have the same qualitative features as with no uncertainty: they have a component that depends on the performance of a manager's own division, the pay-for-performance part, plus a second component, the risk-hedging part, that depends on the cash-flow of the other division. When division cash-flows are positively correlated, incentive contracts display relative-performance compensation; when they are negatively correlated, incentive contracts have cross-pay, that is, an equity component. With respect to the no-uncertainty case, uncertainty increases the cost of incentive provision, with the effect of decreasing pay-for-performance sensitivity and cross-division exposure.

When uncertainty faced by division managers is sufficiently large, uncertainty aversion creates the potential for a significant divergence between beliefs held by division managers and HQ. In this case, HQ find it desirable to hedge division managers' uncertainty by offering compensation contracts with greater cross-division exposure, but at the cost of greater risk. By hedging uncertainty, HQ induce division managers to hold more favorable expectations on their divisions, with a positive impact on effort. Improvement of division managers beliefs also lowers the disagreement discount and the cost of incentive provision. Interestingly, optimal contracts have cross-division exposure even with uncorrelated cash-flows, a sharp contrast with the informativeness principle in principal-agent problems with no uncertainty.[7]

When HQ are uncertainty averse as well, their beliefs are also determined endogenously. HQ uncertainty aversion introduces an additional source of disagreement with division managers making it costlier to offer incentive contracts with relative performance. This happens because relative performance pay essentially involves division managers holding a "short" position in the other division, while HQ hold a "long" position in both divisions, exacerbating the disagreement discounts. The overall effect is to make relative performance contracts costlier and equity-based pay more

[6] Building on Chen and Epstein (2002), in a Appendix A we show that the solution to a corresponding dynamic problem with continuous effort and stationary IID uncertainty is indeed characterized by the solution to the static problem we study in our paper (generalizing the aggregation and linearity results of Holmström and Milgrom, 1987).

[7] More generally, our paper implies that increasing exposure to an uncertain random variables (such as, for example, industry sector or other benchmarks) may offer benefits due to uncertainty hedging.

desirable. Interestingly, pure equity-based contracts are optimal when uncertainty is sufficiently large, irrespective of the correlation between divisional cash-flows.[8]

Overall, the optimal incentive contract will trade off the relative costs and benefits of hedging both risk and uncertainty. We argue that the presence of uncertainty can raise the cost of hedging division managers' risk, creating a conflict between risk and uncertainty hedging. The presence of sufficiently large uncertainty may lead to incentive contracts that substantially deviate from traditional contracts that hedge risk. Furthermore, incentive contracts that hedge division manager uncertainty may also lead them to greater risk exposure, increasing the cost of incentive provision. The potential tension between hedging risk and uncertainty is a new feature in incentive contract design. An important question is the identification and selection of the specific random variables that are better suited to hedge uncertainty.[9]

Our model suggests that cross pay and aggregate performance measures (such as bonuses and equity) can play an important role to hedge uncertainty of division managers (and, more generally, employees) in firms. We also show that the inclusion of aggregate, firm-wide random variables has the added benefit of coordinating internal beliefs, facilitating the formation of a "shared view" in the organization. Additional variables that may be used to hedge uncertainty include industry-wide performance measures (among others). The identification of specific random variables suited to trade-off risk and uncertainty hedging motifs will, in general, depend on the specific exposure to risk and uncertainty of individual firms, and we leave this interesting question for future research.

Our paper offers several novel implications that help explain empirical regularities that are difficult to explain on the basis of risk aversion only. First and foremost, uncertainty aversion can explain the beneficial role of employee bonuses geared to the performance of the entire firm (or one of its larger subdivisions), rather than more narrowly defined performance measures. Second, it can explain the more infrequent use of relative performance compensation and benchmarking, despite their well established benefits within traditional risk aversion. Third, it can explain compensation

[8]Fleckinger (2012) shows that the benefit of relative performance in incentive pay may depend on the impact of effort on the correlation in outcomes. In our paper, correlation is not affected by effort. DeMarzo and Kaniel (2022) argue that relative-performance compensation is not desirable when division managers have "keep-up-with-the-Joneses" preferences.

[9]For example, including exposure in incentive constracts to, say, the result of the Super Bowl may provide little or no value in hedging uncertainty relative to its added risk exposure. Importantly, Section 4 shows that it would be expensive for uncertainty-averse firms to provide to their employees side bets (the two parties take opposite side of a random event), making it more difficult to meet the participation constraint.

practices in business groups, whereby compensation depends on the performance of the entire group, in addition to performance of individual units.[10] Finally, our approach provides a framework for belief formation in organizations that can explain the (endogenous) more optimistic aptitude toward firm future performance for managers in higher position in the corporate hierarchy with respect to rank-and-file employees.[11]

Our paper is linked to several streams of literature. The first one is the traditional principal-agent theory and the theory of optimal contract design within organizations, building on the seminal work by Mirrlees (1975), (1999) and (1976), Holmström (1979), (1982), and Grossman and Hart (1983). Incentive contracts tailored to shareholder value, such as equity, are shown to be optimal when agents can choose their hidden action from rich sets of possible action-profiles (see, for example, Diamond, 1998, and Chassang, 2013). Oyer (2004) suggests that equity-based compensation (for example, through stock-option plans) have the advantage of directly adjusting employees' compensation to their outside options (which may be correlated to firm value), facilitating satisfaction of the participation constraints.

The second stream is the emerging literature on contract theory under uncertainty. Lee and Rajan (2020) study the optimal incentive contract between a principal and a single agent where both parties are uncertainty-averse but risk-neutral and the source of uncertainty is the exact probability distribution of the random cash-flow. The paper shows that, contrary to basic case of uncertainty-neutrality of Innes (1990), the optimal contract has equity-like components. Szydlowski and Yoon (2022) considers a dynamic contracting model where an uncertainty-averse principal designs an optimal (dynamic) contract for an uncertainty-neutral agent, and the source of uncertainty is the agent's cost of effort. Different from our paper, uncertainty leads principals to increase pay-for-performance sensitivity (to preserve incentives under the worst-case scenario). Miao and Rivera (2016) consider the optimal contract between uncertainty-averse principal and an uncertainty- and risk-neutral agent and shows that the principal's preference for robustness can cause the incentive-

[10] For example, the compensation of mutual fund managers depends not only on the performance of their funds, but also on the performance of the entire family of funds, implying a positive cross-fund exposure (see Ma, Tang, and Gomez, 2019). However, the majority of funds are exposed to shared macroeconomic risk, suggesting a positive correlation. Similar practices are common in the investment bank industry.

[11] In Goel and Thakor (2008), greater optimism of senior management depends on (equilibrium) selection of agents with heterogeneous beliefs. In contrast, in our model differences in beliefs emerges endogenously among otherwise identical agents as the outcome of differences in their contractual and hierarchical exposure.

compatibility constraint to be lax.[12] In these papers, similar to ours, disagreement between principal and agent arises as the outcome of the difference in their attitude toward uncertainty. In contrast, in our paper agents are both risk- and uncertainty-averse creating a new tension between hedging their risk and uncertainty exposure through incentive contracts. When agents are both risk and uncertainty averse, hedging uncertainty can interact with hedging risk, and the two goals can conflict with each other. When uncertainty is sufficiently large, the uncertainty-hedging motive can overcome the risk-hedging motive, reversing important properties of optimal incentive contracts absent uncertainty concerns.

Closer to our paper, Sung (2022) considers a model where both principal and the agent are uncertain on both the mean and volatility of the technology controlled by the agent. The paper shows that, consistent with common practice, optimal incentive contracts include exposure to underlying volatility. Different from our paper, exposure to uncertain volatility allows principals to design optimal contracts that achieve agreement with the agent. Kellner (2015) examines a principal-agent model with multiple agents and moral hazard, where the principal is risk and uncertainty neutral; agents can be risk and uncertainty averse and uncertainty is modeled as smooth ambiguity (Klibanoff et al., 2005). Because agents are exposed to the same source of uncertainty in this setting, Kellner (2015) shows tournament incentives are optimal with sufficient uncertainty.

In Carroll (2015) a risk-neutral principal, who is uncertain about the set of actions available to a risk- and uncertainty-neutral agent, optimally grants the agent a linear contract that aligns their payoffs. Linear (or affine) contracts are optimal robust contracts under very weak assumptions on the source of uncertainty characterizing the set of technologies available to the agent.[13] In the spirit of Holmström (1982), Dai and Toikka (2022) examines a moral hazard in teams problem, where a risk-neutral principal designs contracts that are robust to uncertainty regarding the underlying game played by uncertainty-neutral agents. The paper shows that optimal robust contracts must have the property that agents' compensation covaries positively, and provides conditions under which optimal robust contracts are linear (or affine). Finally, Walton and Carroll (2022) show that, under mild conditions, optimal contracts are linear within several possible configurations of the organization structure, when principal are risk neutral and agents are risk and uncertainty neutral.

[12] Lee and Rivera (2021) consider optimal liquidity management under ambiguity.

[13] Carroll and Meng (2016) give a microfoundation of uncertainty with linear contracts.

The paper is organized as follows. The general contracting problem is presented in Section 1. Section 2 examines the impact of contracts on beliefs and effort under uncertainty. Optimal incentive contracts by uncertainty-neutral HQ are derived in Section 3, and by uncertainty-averse HQ in Section 4. In Section 5 we discuss robustness of our results to alternative specification of beliefs. The impact of uncertainty aversion on organizational beliefs discussed in Section 6. Section 7 presents empirical implications of our paper. Section 8 concludes with directions for further research. All proofs are in the Technical (online) Appendix.

1 Uncertainty and Contracting

1.1 The Basic Model

We consider a firm composed by two divisions (or business units) denoted by $d \in \{A, B\}$.[14] Each division is run by a division manager supervised by HQ. At the beginning of the period, $t = 0$, each division manager chooses effort, $a_d \in \mathbb{R}_+$, affecting the probability distribution of their divisional cash-flow, Y_d, realized at the end of the period $t = 1$. We assume that the cash flows $Y \equiv (Y_A, Y_B)$ have a joint normal distribution $N(\mu, \Sigma)$ with mean $\mu \equiv (\mu_A, \mu_B)$ and variance-covariance matrix Σ. Managerial effort affect the means of the distributions, and we set $\mu_d = a_d q_d$, where q_d represents the productivity of division $d \in \{A, B\}$. Division cash-flows Y_d are homoskedastic, with variance σ^2, and may be (positively or negatively) correlated, with correlation coefficient ρ; we assume that effort does not affect the variance-covariance matrix, Σ.[15]

Exerting effort is costly: each division manager suffers a monetary cost $c_d(a_d)$, where $c_d : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable, increasing and convex function. For analytical tractability, we set $c_d(a_d) = \frac{1}{2Z_d} a_d^2$, where Z_d characterizes effort efficiency of division managers. Division managers have preferences with constant absolute risk aversion (CARA), while HQ are assumed to be risk neutral, for simplicity.

Effort exerted by a division manager is not observable by either HQ or the other division manager, creating moral hazard. To promote effort, HQ offer division managers incentive contracts, $w \equiv \{w_d\}_{d \in \{A,B\}}$. Given a pair of incentive contracts w, division manager $d \in \{A, B\}$ earns an end-of-period payoff $U(w_d) = -e^{-r w_d}$, where r is the coefficient of absolute risk aversion, which we assume to be the same for both divisional managers.

[14] Our model can equivalently be interpreted a describing separate divisions of a company (such as, for example, a conglomerate), or separate business units of a "pure-play" firm.

[15] Hemmer (2017) and Ball et al. (2020) study contracts when effort affects Σ.

The game unfolds as follows. At the beginning of the period, $t = 0$, HQ choose incentive contracts w_d for each division manager $d \in \{A, B\}$; HQ can commit to contracts $\{w_d\}_{d \in \{A,B\}}$, which are observable to both division managers. After incentive contracts are offered and accepted, division managers simultaneously choose their level of effort, a_d. At the end of the period, $t = 1$, division managers are compensated according to the realized output Y, and consumption takes place.

1.2 Uncertainty aversion

Contrary to the standard principal-agent paradigm of Holmström (1979), we assume both HQ and division managers are uncertain on the exact probability distribution of the end-of-period cash flows for each division. Specifically, we assume that division managers and HQ are uncertain on division managers' productivity, q_d. The presence of such uncertainty affects the (perceived) marginal productivity of effort and, thus, division managers' incentives to exert effort.

Following Miao and Rivera (2016) and Dicks and Fulghieri (2019) and (2021), we model uncertainty (or "ambiguity") aversion by adopting the minimum expected utility (MEU) approach of Gilboa and Schmeidler (1989) and Chen and Epstein (2002). A key feature of this approach is that agents do not have a single prior on future events but, rather, they believe that the probability distribution on future events belongs to a certain set, $\mathcal{P}$, denoted the "core-beliefs set," and maximize their minimum expected utility

$$U^a = \min_{p \in \mathcal{P}} E_p \left[U \left(w \right) \right], \tag{1}$$

where p is a probability distribution, and U is a von Neumann-Morgenstern utility function. The key feature of the MEU approach to uncertainty aversion is that beliefs are endogenous, as determined by the minimum expected utility criterion. An important implication is that uncertainty-averse agents weakly prefer randomizations over random variables (more precisely, over acts as described in Anscombe and Aumann 1963) rather than each individual variable in isolation. This property is a direct consequence of the uncertainty-aversion axiom of Gilboa and Schmeidler (1989) and is known as "uncertainty hedging."

The benefits of uncertainty hedging under uncertainty aversion are analogous to the traditional benefits of diversification under risk aversion. Intuitively, this feature can immediately be seen by noting that, given two random variables, y_j, $j \in \{1, 2\}$, with joint distribution $p \in \mathcal{P}$, by the minimum operator, we have that

$$\zeta \min_{p\in\mathcal{P}} E_p\left[U\left(y_1\right)\right] + (1-\zeta) \min_{p\in\mathcal{P}} E_p\left[U\left(y_2\right)\right] \leq \min_{p\in\mathcal{P}} \{\zeta E_p\left[U\left(y_1\right)\right] + (1-\zeta) E_p\left[U\left(y_2\right)\right]\} \tag{2}$$

for all $\zeta \in [0,1]$. The key driver of our paper is that condition (2) can hold as a strict inequality.

In the spirit of Hansen and Sargent (2001, 2008), we model the core-beliefs set $\mathcal{P}$ by using the notion of relative entropy. For a given pair of distributions $\hat{P}$ and $P,$ with corresponding densities $\hat{p}$ and p, defined on the same probability space, the relative entropy of $\hat{P}$ with respect to P is the Kullback-Leibler divergence of $\hat{P}(x)$ with respect to $P(x)$, namely

$$R\left(\hat{P}(x)|P(x)\right) \equiv \int \hat{p}(x) \ln\left(\frac{\hat{p}(x)}{p(x)}\right) dx. \tag{3}$$

The core-belief set for uncertainty-averse agents is then defined as

$$\mathcal{P}(P(x)) \equiv \{\hat{P} : R\left(\hat{P}(x)|P(x)\right) \leq \eta_P\}. \tag{4}$$

where P represents a given reference probability and $\hat{P}(x)$ is an admissible belief held by the agent. From (3), it is easy to see that the relative entropy of $\hat{P}(x)$ with respect to P represents the (expected) log-likelihood ratio of P, when the "true" probability distribution is $\hat{P}(x)$. Intuitively, the core-beliefs set $\mathcal{P}(P)$ can be interpreted as the set of probability distributions, $\hat{P}(x)$, that, if true, would not reject the ("null") hypothesis P in a (log) likelihood-ratio test. The distribution P can be interpreted as characterizing an agent's "view" about the true probability $\hat{P}(x)$, where the parameter η_P represents the degree of confidence on P. A small value of η_P represents situations where agents have more confidence that the distribution P, while a large value of η_P corresponds to situations where there is great uncertainty.[16] Intuitively, the relative entropy approach considers as admissible only beliefs that are not "too unlikely" to be the true probability distribution, given the reference probability. The effect is restrict the core-beliefs set by excluding as implausible probability distributions that give too much weight to extreme events, effectively "trimming" agents' pessimism.

Because in our model agents view as uncertain the productivity q_d of the two divisions, we denote the set $\hat{P}$ of beliefs held by agent $i \in \{HQ, A, B\}$ on division productivity as $\hat{q}^i \equiv (\hat{q}^i_A, \hat{q}^i_B) \in$

[16] As in Hansen and Sargent (2001, 2008), relative entropy characterizes the extent of "misspecification error" when agents believe that the model is P^a when the true model is P.

$K^i(\hat{q}^i|q^i)$, where $\hat{q}^i_d$ represents the belief held by agent i on the productivity of division d, and q^i_d is the corresponding reference belief. A key property of relative entropy, and one which plays the crucial role in our paper, is that under the relative entropy criterion the core-beliefs set $K^i(\hat{q}^i|q^i)$ is a strictly convex set with smooth boundaries.[17] This property allows (2) to hold has a strict inequality, making uncertainty hedging valuable. Intuitively, when division managers are exposed only to their own division uncertainty, they will be concerned about facing the lowest possible level of division productivity, with a negative effect on effort. In contrast, when division managers are exposed to uncertainty from both divisions, they will regard the possibility that both divisions are characterized by extreme levels of productivity sufficiently unlikely to be ruled out by the relative entropy criterion, excluding it from the (4) with a beneficial effect on effort.[18] This implies that, by proper design of incentive contracts, HQ can affect the probability measure used by division managers to assess the productivity of their division, mitigating the adverse effect of uncertainty on effort. In Section 5, we will discuss the case the limiting case of the core belief set is rectangular.[19]

Finally, at times, we will assume that divisions are symmetric:

$$(S) : Z_A = Z_B \equiv Z, \; q_A = q_B \equiv q, \text{ and } K_i(q_i) = K(q) \text{ for } i \in \{HQ, A, B\}. \tag{5}$$

1.3 The optimal contracting problem

At the beginning of the period, $t = 0$, company HQ offers to division managers contracts $w_d(Y)$, $d \in \{A, B\}$, which may depend on realized output of both divisions. For ease of exposition, we restrict our analysis to the case of affine incentive contracts. In Appendix A, we study a dynamic, stochastic continuous-time version of our model with IID ambiguity as in Chen and Epstein (2002). In the spirit of Holmström and Milgrom (1987), Theorem A1 in Appendix A shows that the solution to the dynamic model is characterized by the solution of a corresponding static problem where HQ offer only affine contracts that depend on end-of-period cash flows, as considered in the main body

[17] For a general discussion, see Theorem 2.5.3 and 2.7.2 of Cover and Thomas (2006). The main results of our paper depend only on the property that the core belief sets are strictly convex sets. In addition to core-belief sets based on relative entropy, this property is shared by core-belief sets defined by divergences that are strictly monotonic and continuous.

[18] In Section 3 we will show that division managers owning a positive ("long") exposure to the other division cash flow (due to an equity component in incentive pay) will be concerned about the other division having low productivity, reducing the value of their own equity stake. Conversely, division managers with a negative (or "short") position in the other division (due to relative performance component in incentive pay) will be concerned about the other division having high productivity, again reducing the value of their equity stake.

[19] The case of rectangular core belief set is discussed, for example, in Chen and Epstein (2002)

of the paper.

Given affine incentive contract, we set

$$w_d(Y) = s_d + \beta_d Y_d + \gamma_d Y_{d'},$$

where we can interpret the fixed component, s_d, as a "base pay" and the variable component as the "incentive pay." The incentive pay for division managers may be composed of two parts. The first part is the "pay-for-performance" component," which depends on the realized output of their own division Y_d, and where the coefficient β_d represents the pay-for-performance sensitivity. The second part is a "cross-pay" exposure, which depends on realized output of the other division, $Y_{d'}$, where setting $\gamma_d > 0$ represents an equity-based component in compensation, and setting $\gamma_d > 0$ makes compensation to depend on the relative performance of the two divisions.

Given CARA utility functions for division managers, we can write the HQ problem in certainty equivalent form as follows. Let $\hat{q}^d \equiv (\hat{q}_A^d, \hat{q}_B^d)$ represent the beliefs held by division manager d on the productivity of both divisions, and let $\hat{q}^{HQ} \equiv (\hat{q}_A^{HQ}, \hat{q}_B^{HQ})$ the corresponding beliefs held by HQ. Given an incentive contract w_d, a division manager utility function in certainty equivalent form is

$$u_d(\hat{q}^d, a) \equiv E\left[w_d | \hat{q}^d, a\right] - \frac{r}{2} Var(w_d) - c_d\left(a_d\right), \tag{6}$$

where

$$Var(w_d) = \sigma^2\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right)$$

is the variance of incentive pay. Note that the expected value of the incentive pay to division managers, $E\left[w_d(Y)|\hat{q}^d, a\right]$, depends on both their beliefs on the productivity of their own division, $\hat{q}_d^d$, through the pay-for-performance component, and on the productivity of the other division, $\hat{q}_{d'}^d$, through the cross-pay component. Similarly, that the expected value of the incentive pay, will also depend on the level of effort exerted by the two division managers. In contrast, because agent view as uncertain only division productivity and effort does not affect the variance-covariance matrix Σ, the term $Var(w_d)$ does not depend on a division manager beliefs and effort levels.

The problem for HQ is to choose a pair of incentive contracts and action profiles, $\{w_d, a_d\}_{d\in\{A,B\}}$, that solves

$$\max_{\{w,a\}} \quad \min_{\hat{q}^{HQ} \in K_{HQ}(a)} \pi(\hat{q}^{HQ}) \equiv \sum_{d\in\{A,B\}} E\left[Y_d(a_d) - w_d\left(Y\right)|\hat{q}^{HQ}\right] \tag{7}$$

subject to the constraint that division managers maximize the certainty equivalent of their objective function

$$\max_{\tilde{a}_d} \min_{\hat{q}^d \in K_d(a)} u_d(\hat{q}^d, a) \equiv E\left[w_d | \hat{q}^d, a\right] - \frac{r}{2} Var(w_d) - c_d\left(a_d\right), \tag{8}$$

and to the participation constraints

$$\min_{\hat{q}^d \in K_d(a)} u_d(\hat{q}^d, a) \geq u_0 = 0 \tag{9}$$

for $d, d' \in \{A, B\}$, and $d \neq d'$, where u_0 is a division manager's reservation utility, normalized to zero.

Note that in problem (7) - (9) a division manager's uncertainty exposure is endogenous and is determined by the incentive contract, w_d, offered by HQ. Contractual exposure concurs to determine a division manager's beliefs, $\hat{q}^d$. Given their higher-level position in the firm hierarchy, HQ exposure to uncertainty is determined by their residual claim in firm cash-flow, given incentive contracts offered to both division managers in the firm.[20] HQ hierarchical exposure concurs to determine HQ beliefs, $\hat{q}^{HQ}$. The triplet $\{\hat{q}^{HQ}, \hat{q}^A, \hat{q}^B\}$ determines the belief structure prevalent in the firm.

Definition 1 *An equilibrium is a set of contracts , $w = \{w_d\}_{d \in \{A,B\}}$, and action profile $\{a_A, a_B\}$, such that:*

(i) Given incentive contracts w, each division managers selects effort, a_d, optimally, solving (8), given the other division manager's action, $a_{d'}$ for $d' \neq d$;

(ii) HQ offer contracts w that maximizes (7) subject to (8) - (9)

The main trade-offs faced by HQ in problem (7) - (9) can be decomposed as follows. Because of translation invariance of CARA, the fixed component of incentive contracts, s_d, is set to make the participation constraint (9) bind in optimal contracts, giving

$$s_d = c_d\left(a_d\right) + \frac{r}{2} Var\left[w_d\left(Y\right)\right] - E\left[w_d | \hat{q}^d, a\right].$$

[20]We assume HQ are full residual claimants in firm cash-flow. More generally, HQ act in the context of incentive contracts from a compensation committee, exposing them to contractual exposure as well.

after substitution into the objective function (7), we obtain

$$\pi(\hat{q}^{HQ}) = \sum_{d\in\{A,B\}} \left[E(Y_d(a_d)|\hat{q}_d^{HQ}) - \frac{r}{2} Var\left[w_d\left(Y\right)\right] - c_d\left(a_d\right) - \left(E\left[w_d|\hat{q}_d^d, a\right] - E[w_d|\hat{q}_d^{HQ}, a] \right) \right]. \tag{10}$$

HQ payoff consists of four components. The first one is the expected value of the two divisions, which depends on effort exerted by division managers, a; the second one is given by the required risk premia for division managers, $Var\left[w_d\left(Y\right)\right]$; the third one is the cost of providing effort by division managers, $c_d\left(a_d\right)$. These components are common to the traditional problem without uncertainty aversion. The last component is new and is due to uncertainty aversion, and is discussed below.

Uncertainty aversion affects HQ payoff in three separate ways. First, HQ valuation of both divisions, $E(Y_d(a_d)|\hat{q}_d^{HQ})$, is based on beliefs they hold, $\hat{q}^{HQ}$, which are endogenous. Second, from the incentive constraint (8), effort exerted by division managers depends on their "worst-case" scenario, $\hat{q}^d$, negatively affecting effort. This implies that HQ must increase the pay-for-performance sensitivity, β_d, to elicit any desired level of effort, increasing the cost of incentive provision. The worst-case scenario, $\hat{q}^d$, however, is itself endogenous, and depends on a division manager's overall exposure to uncertainty through incentive contract, $w_d\left(Y\right)$. A key feature of our paper is that, by hedging uncertainty through incentive contracts, HQ can improve a division manager's assessment of her division productivity, $\hat{q}_d^d$, promoting effort.

The third effect of uncertainty aversion, the last term in (10), is to create a divergence between HQ and division managers on the valuation of compensation contracts, $E\left[w_d|\hat{q}_d^d, a\right] - E[w_d|\hat{q}_d^{HQ}, a]$. This terms acts through division managers' participation constraints (9), and reflects the fact that HQ value compensation contracts at their own worst-case scenario, $\hat{q}^{HQ}$, while division managers value contracts at theirs, $\hat{q}^d$, creating a disagreement on the assessment of the value of an incentive contract to division managers and its cost to HQ. In particular, if HQ are more confident than division managers on their division productivity, $\hat{q}_d^{HQ} > \hat{q}_d^d$, division managers value their compensation contracts at a discount relative to HQ valuation, making it more costly (from HQ point of view) to satisfy their incentive and participation constraint, (8)-(9). We denote this additional cost of incentive-based pay as a "disagreement discount," which represents the "Knightian" cost of disagreement.

2 Uncertainty and Incentive Contracts

As a benchmark, we first characterize the solution to the optimal contracting problem for our two-division firm without uncertainty, a setting similar to Holmström and Milgrom (1987).

2.1 The No-Uncertainty Benchmark

Absent uncertainty concerns, with $K_{HQ} = K_d = \{0\}$, HQ and division managers share the same beliefs and agree on the reference probability P^a.

Theorem 1 *(Holmström and Milgrom) Let HQ be risk neutral: optimal contracts are linear functions of the end-of-period cash-flows of both divisions:* $w_d(h_1) = s_d + \beta_d Y_{d,1} + \gamma_d Y_{d',1}$, *for all* t *and* $d \in \{A, B\}$, *with*

$$\beta_d = \frac{1}{1 + r\sigma^2 (1-\rho^2) / \left(Z_d q_d^2\right)}, \quad \gamma_d = -\rho\beta_d, \tag{11}$$

and induce optimal effort

$$a_d = \beta_d Z_d q_d = \frac{Z_d q_d}{1 + r\sigma^2 (1-\rho^2) / \left(Z_d q_d^2\right)}. \tag{12}$$

Under universal risk neutral, $r = 0$, the optimal contract makes division managers residual claimants, $\beta_d = 1$, leading to first-best effort; cross-pay γ_d is indeterminate because side bets are irrelevant for risk-neutral agents. The presence of risk aversion increases the cost of incentive provision and reduces the pay-for-performance sensitivity, due to term $r\sigma^2 \left(1-\rho^2\right) / \left(Z_d q_d^2\right)$. Optimal contract depends now on the correlation of end-of-period cash-flows of both divisions. If cash-flows are correlated, it is optimal for HQ to hedge division manager risk exposure. With positive correlation, HQ set $\gamma_d < 0$ and contracts display "relative-performance" compensation; with negative correlation, HQ set $\gamma_d > 0$ and incentive contracts display an equity component through cross-pay. Hedging division manager risk exposure reduces the cost of incentive provision and allows HQ to increase pay-for-performance sensitivity, improving incentives. When cash-flows are uncorrelated, cross-pay generates only incremental risk exposure with no risk-hedging benefit, and optimal contracts set $\gamma_d = 0$ (the "informativeness principle").

2.2 Incentive contracts and beliefs

For tractability, and to generate closed-form solutions, similar to Dicks and Fulghieri (2019) and (2021) we consider a parametric approximation of the core-belief set (4).[21] Specifically, we assume

[21] In our model, tractable closed-form solutions would be possible only for the case of uncertainty-neutral HQ.

that HQ and division managers consider beliefs $\hat{q}^i$ in a neighborhood of the reference probability implied by the pair $q = (q_A, q_B)$, as follows. Define $\chi_d^i = \left|\frac{\hat{q}_d^i - q_d}{q_d}\right|$ as the relative error of player i about division d and the distance measure $D\left(\chi_d^i\right) = -\log\left(1 - \chi_d^i\right)$. We denote the core-belief set for agent i as

$$K^i(\hat{q}^i|q^i) \equiv \left\{\hat{q}^i|D\left(\chi_A^i\right) + D\left(\chi_B^i\right) \leq \eta^i\right\}, \tag{13}$$

for $i \in \{HQ, A, B\}$.[22] The set described in (13) is plotted in Figure 1 on page 40, with relative entropy for comparison.

We start with the characterization of division managers' belief assessments on the productivity of both divisions, which depend on the pair of incentive contracts offered by HQ. From (6), given incentive contract $w = \{w_d\}_{d \in \{A,B\}}$, division managers beliefs $\hat{q}^d(a, w)$ solve

$$\begin{aligned} \arg\min_{\hat{q}_d} \; u_d(\hat{q}_d) &= E\left[w_d|\hat{q}^d, a\right] - \frac{r}{2}Var(w_d) - c_d\left(a_d\right), \qquad (14) \\ s.t. \quad & \ln\left(\frac{1}{1 - \left|\frac{\hat{q}_A - q_A}{q_A}\right|}\right) + \ln\left(\frac{1}{1 - \left|\frac{\hat{q}_B - q_B}{q_B}\right|}\right) \leq \eta_d, \end{aligned}$$

for $d \in \{A, B\}$. Note that incentive contracts offered by HQ will have $\beta_d > 0$, so that division managers will exert strictly positive effort, $a_d > 0$.

Lemma 1 *Let* $\beta_d a_d > 0$ *and*

$$H_d \equiv \frac{\gamma_d a_{d'} q_{d'}}{\beta_d a_d q_d}. \tag{15}$$

A division manager's assessment of the productivity of both divisions, $\hat{q}^d(w_d) = \{\hat{q}_d^d, \hat{q}_{d'}^d\}$*, for* $d, d' \in \{A, B\}$*, and* $d \neq d'$*, depends on her contractual exposure,* w_d*, and is equal to:*

i) $\hat{q}_d^d = q_d$*, and* $\hat{q}_{d'}^d = e^{-\eta_d} q_{d'}$ *for* $H_d \geq e^{\eta_d}$

ii) $\hat{q}_d^d = (e^{-\eta_d} H_d)^{\frac{1}{2}} q_d$ *and* $\hat{q}_{d'}^d = \left(e^{-\eta_d}\frac{1}{H_d}\right)^{\frac{1}{2}} q_{d'}$ *for* $H_d \in (e^{-\eta_d}, e^{\eta_d})$

iii) $\hat{q}_d^d = e^{-\eta_d} q_d$ *and* $\hat{q}_{d'}^d = q_{d'}$ *for* $H_d \in [-e^{-\eta_d}, e^{-\eta_d}]$

iv) $\hat{q}_d^d = (e^{-\eta_d}|H_d|)^{\frac{1}{2}} q_d$ *and* $\hat{q}_{d'}^d = \left[2 - \left(e^{-\eta_d}\frac{1}{|H_d|}\right)^{\frac{1}{2}}\right] q_{d'}$ *for* $H_d \in (-e^{\eta_d}, -e^{-\eta_d})$

v) $\hat{q}_d^d = q_d$ *and* $\hat{q}_{d'}^d = (2 - e^{-\eta_d}) q_{d'}$ *for* $H_d \leq -e^{\eta_d}$

Division managers beliefs toward division productivity depend on the relative exposure to the

[22] Note that this characterization of the core-beliefs set allows a great degree of tractability: when an economic agent has sufficient positive exposure to both divisions, so that $\hat{q}_d^i < q_d$, the minimization problem is isomorphic to the cost minimization problem with Cobb-Douglass utility. Further, the set is symmetric around $q = (q_A, q_B)$, making uncertainty hedging neutral with respect to positive or negative exposure to cross-division uncertainty.

cash-flow from each division, measured by H_d, as affected by incentive contract w_d. Because H_d affects the relative exposure to uncertainty of the two divisions, we refer to H_d as the "uncertainty hedging" ratio. Note that $sign(H_d) = sign(\gamma_d)$ and that H_d is an increasing function of γ_d.

Several features emerge from Lemma 1. When HQ grant pay-for-performance only, $\gamma_d = 0 = H_d$, or a small exposure to the other division cash-flow, as in case (iii), division managers will assess the prospects of their own division conservatively, with $\hat{q}_d^d = e^{-\eta} q_d$, and they are less confident on their own division productivity, disincentivizing effort.

Division manager assessments of productivity of their own division, $\hat{q}_d^d$, is however an increasing function of their exposure to the other division, $|\gamma_d|$. Thus, incentive contracts that offer progressively increasing exposure to the other division, as in case (ii) and (iv), induce division managers to become more confident on their own division, $\hat{q}_d^d$. Finally, if incentive contracts offer significant exposure to other division, a large value of $|\gamma_d|$, as in case (i) and (v), division managers will become very confident on their own division, setting $\hat{q}_d^d = q_d$. This beneficial effect on division manager beliefs can be obtained with either cross-pay, $\gamma_d > 0$, or relative-performance, $\gamma_d < 0$.

The impact of $|\gamma_d|$ on a division manager's assessment of the productivity of the other division depends on the sign of γ_d. If the incentive contract includes cross-pay, $\gamma_d > 0$, increasing exposure to the other division progressively worsens the assessment of that other division productivity, as in cases (ii) and (i). If the incentive contract includes relative performance, $\gamma_d < 0$, increasing exposure to the other division (lower γ_d) progressively improves the assessment of its productivity, as in cases (iv) and (v), where in both cases $\hat{q}_{d'}^d > q_{d'}$. The more optimistic assessment reflects the fact that, when $\gamma_d < 0$, better performance in the other division reduces a division manager's compensation.

2.3 Incentive contracts and effort

Given division managers' beliefs, characterized in Lemma (1), effort is determined by solving

$$\max_{a_d} \; u_d(a, \hat{q}^d(a, w)) = E\left[w_d | \hat{q}^d, a_d, a_{d'}\right] - \frac{r}{2} Var(w_d) - c_d\left(a_d\right), \tag{16}$$

for $d \in \{A, B\}$ and $d \neq d'$. The Nash equilibrium of effort selection by division managers is determined in the following.

Lemma 2 *Given incentive contracts, $\{w_d = (\beta_d, \gamma_d)\}_{d \in \{A,B\}}$, there is a unique Nash equilibrium* *effort,* $\{a_A, a_B\}$ *where* $a_d = \beta_d Z_d \hat{q}_d^d$ *and division manager beliefs,* $\hat{q}_d^d$*, are as in Lemma 1. Equilibrium effort* a_d *is increasing in pay-performance sensitivity,* β_d*, exposure to the other division,* $|\gamma_d|$*, efficiency of effort,* Z_d*, and decreasing in uncertainty* η_d*. Further, if* $|H_d| \in (e^{-\eta_d}, e^{\eta_d})$*,* a_d *is also increasing in* $\beta_{d'}$*,* $|\gamma_{d'}|$*, and* $Z_{d'}$*, and decreasing in* $\eta_{d'}$*.*

Lemma 1 and 2 imply that incentive contracts affect division manager effort through two distinct channels. The first is the traditional effect of inducing effort by rewarding division managers on the basis of direct performance measures, and captured by β_d. The second channel is through the impact of incentive contracts on managerial assessment of the success probability of their projects, $\hat{q}^d(w_d)$. This implies that HQ can use incentive contract design to lead uncertainty-averse division managers to hold more favorable assessment of the productivity of their own division, with positive effect on effort. This channel due to uncertainty hedging is the key driver of our paper.

If division managers are uncertainty neutral, their optimal level of effort in (12), a_d, is increasing in her own division-based pay, β_d, but is not affected by either their cross-division pay, γ_d, nor the action of the other division manager, $a_{d'}$. The only effect of cross-division exposure is to hedge a division manager's risk exposure, reducing the cost of incentive provision. In contrast, if division managers are uncertainty averse, the presence of cross pay, $|\gamma_d| \neq 0$, reduces the relative exposure of division managers to the uncertainty on their division, potentially improving the assessment of their division productivity.

Note also that uncertainty aversion introduces a strategic complementarity across division managers' effort. From Lemma 1, exposure to the other division, $|\gamma_d| > 0$, makes effort exerted by a division manager, a_d, increasing in effort of the other division manager, $a_{d'}$. Greater effort from the other manager decreases the relative exposure of a division manager to uncertainty on her own division, leading to more favorable beliefs and greater effort. This new source of externality is due to uncertainty hedging, and is driven solely by beliefs.

We examine two possible configurations of HQ beliefs: uncertainty neutrality and uncertainty aversion. Because uncertainty-neutral HQ hold firm beliefs on division productivity, we can denote this case as one of a "visionary leadership." In contrast, uncertainty-averse HQ pragmatically adapt (in equilibrium) their beliefs to firm characteristics, we can denote this case as one of "pragmatic leadership."

3 Uncertainty-Neutral Principal

When HQ are uncertainty neutral, they hold beliefs $\hat{q}_d^{HQ} = q_d$ for both divisions. Given Lemma A1, problem (A12)-(A14) becomes

$$\max_{\{w_d, a_d\}_{d\in\{A,B\}}} \sum_{d\in\{A,B\}} E\left[Y_d(a_d) - w_d\left(Y\right)|q_d\right] \tag{17}$$

subject to the incentive and participation constraints

$$\max_{a_d} \min_{\hat{q}_d} \; u_d = E\left[w_d|\hat{q}^d, a_d, a_{d'}\right] - \frac{r}{2}Var(w_d) - c_d\left(a_d\right), \tag{18}$$

$$\min_{\hat{q}_d} \; u_d = E\left[w_d|\hat{q}^d, a_d, a_{d'}\right] - \frac{r}{2}Var(w_d) - c_d\left(a_d\right) \geq 0, \tag{19}$$

for $d \in \{A, B\}$ and $d \neq d'$. To separate the effect of uncertainty and risk aversion, we consider first the case in which division managers are uncertainty averse but risk-neutral, allowing us to identify more clearly their respective role in optimal contract design.

Theorem 2 *If HQ are risk- and uncertainty-neutral and division managers are uncertainty averse but risk neutral, optimal incentive contracts have*

$$|H_d| = \frac{|\gamma_d| a_{d'} q_{d'}}{\beta_d a_d q_d} = 1,$$

which induce division managers beliefs $(\hat{q}_d^d, \hat{q}_{d'}^d)$ *to be equal to*

$$\hat{q}_d^d = e^{-\frac{\eta_d}{2}} q_d < q_d, \tag{20}$$

$$\hat{q}_{d'}^d = e^{-\frac{\eta_d}{2}} q_{d'} < q_{d'} \text{ for } \gamma > 0 \text{ and } \hat{q}_{d'}^d = \left(2 - e^{-\frac{\eta_d}{2}}\right) q_{d'} > q_{d'} > \hat{q}_d^d \text{ for } \gamma < 0$$

for $d, d' \in \{A, B\}$, *and* $d \neq d'$. *Optimal contracts set*

$$\beta_d = \frac{1}{1 + 3\left(1 - \hat{q}_d^d / q_d\right)} < 1, \text{ and } |\gamma_d| = \xi_d \beta_d, \text{ where} \tag{21}$$

$$\xi_d \equiv \frac{a_d q_d}{a_{d'} q_{d'}} = \frac{1 - 3\left(1 - \hat{q}_{d'}^{d'} / q_{d'}\right)}{1 - 3\left(1 - \hat{q}_d^d / q_d\right)} \frac{\hat{q}_d^d / q_d}{\hat{q}_{d'}^{d'} q_{d'}} \frac{Z_d q_d^2}{Z_{d'} q_{d'}^2}. \tag{22}$$

Pay-for-performance sensitivity, β_d*, and effort,* a_d*, are both decreasing in uncertainty,* η_d*. If condition (S) holds, equity is optimal,* $\beta_d = \gamma_d$*, and* $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\eta}{2}} q < q$.

If division managers are uncertainty averse but risk neutral, hedging risk is not a concern. The presence of uncertainty, by making division managers less confident than HQ on the productivity of their own division, has two adverse effects. First, it has the detrimental effect on the incentives to exert effort by their managers. This implies that HQ must increase pay-for-performance sensitivity to elicit any desired level of effort. Second, more conservative beliefs reduce the value of the incentive contract, w_d, as assessed by division managers, relative to the value assessed by the more confident HQ, making it more expensive for HQ to meet the their participation constraint (the disagreement discount). The combined effect is to make it costlier for HQ to induce effort, leading to a reduction of the pay-for-performance sensitivity β_d in (21). Note that the pay-for-performance sensitivity, β_d, and thus effort, a_d, are both decreasing functions of the extent of the disagreement between a division manager and HQ, represented by the term $\hat{q}_d^d/q_d$ in (22).

The role of cross-division exposure, $|\gamma_d|$, is to improve division managers' beliefs by hedging their uncertainty. From Lemma 1, an increase of cross-division exposure (partially) offsets the negative effect of uncertainty on beliefs, promoting effort. Absent risk-aversion considerations, the optimal contract hedges a division manager exposure to uncertainty by equalizing her exposure to cash-flow uncertainty from each division, setting the uncertainty hedge ratio $|H_d| = 1$.

Note that, because of uncertainty neutrality, HQ are indifferent between granting compensation with cross-pay, $\gamma_d > 0$, or relative-performance, $\gamma_d < 0$, as the optimal contracts depends only on the size of the cross-division exposure, $|\gamma_d|$, and not by its sign. The extent of cross-division exposure, $|\gamma_d|$, is still proportional to the pay-for-performance sensitivity parameter, with $|\gamma_d| = \xi_d \beta_d$, where the proportionality factor ξ_d depends on the relative exposure to uncertainty of the two division managers, affecting the term $\hat{q}_d^d/q_d$, and the relative size of the two divisions, captured by the term $Z_d q_d^2 / Z_{d'} q_{d'}^2$ in (22). This implies that cross division exposure is greater for (relatively) less confident division managers and for larger divisions.

If divisions are symmetric, the uncertainty hedge ratio can be set to unity by use of pure equity contracts: $\beta = \gamma < 1$. Interestingly, in this case, both division managers hold the same beliefs on their own as well as the other division, $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\eta}{2}} q$, leading to consensus (that is, a "shared view") in the organization. Also, HQ hold more optimistic beliefs than division managers, $q > \hat{q}_d^d = e^{-\frac{\eta_d}{2}} q$, making HQ to appear as "visionary" in the organization. Finally, absent risk aversion, a contract with extreme relative performance, with $\gamma = -\beta$, is also optimal. In this case, from (20), we have $\hat{q}_d^d < q < \hat{q}_{d'}^d$, and division managers are more confident on the other division that they are on their own, creating envy and discord in the organization, a potentially undesirable configuration of internal beliefs.

An important implication of Theorem 2 is that the optimal contract (21) differs from the corresponding case of risk-neutral division managers with no uncertainty of Theorem 1, where division managers become full residual claimants in their own division, with $\beta_d = 1$, and with no role for cross-pay γ_d. With uncertainty concerns, making division managers full residual claimant exacerbates pessimism toward their own division, depressing effort. In this case, HQ find it optimal to reduce pay-for-performance sensitivity, $\beta_d < 1$, and to hedge division manager uncertainty by offering exposure to the other division's uncertainty, setting $|\gamma_d| > 0$.

The presence of risk-aversion affects optimal contracts because hedging uncertainty creates a risk exposure, which is costly for risk-averse division managers. Optimal contracts trade off the relative benefits of risk-hedging and uncertainty-hedging motives. For tractability, with risk-averse division managers we focus on the symmetric case, (S).

Theorem 3 *Let condition (S) hold. There is a threshold $\bar{\eta}(r,\rho)$ (defined in Appendix), with $\bar{\eta}(0,\rho) = 0$, such that for $d, d' \in \{A, B\}$, and $d \neq d'$:*

1. If $\eta \leq \bar{\eta}$, optimal incentive contracts induce division managers beliefs $\hat{q}_d^d = e^{-\eta} q$ and $\hat{q}_{d'}^d = q$, by setting

$$\beta = \frac{1}{1 + \left(1 - \hat{q}_d^d / q\right) + r\sigma^2 \left(1 - \rho^2\right) / \left(Z q \hat{q}_d^d\right)} > 0, \quad \gamma = -\rho\beta. \tag{23}$$

Pay-for-performance sensitivity, β, and Nash equilibrium effort, a, are both decreasing in uncertainty, η; the threshold $\bar{\eta}(r,\rho)$ is increasing in r and $|\rho|$.

2. If $\eta > \bar{\eta}$, optimal incentive contracts induce division managers to hold the same beliefs as in (20) of Theorem 2 by setting

$$\beta = \frac{1}{1 + 3\left(1 - \hat{q}_d^d / q\right) + 2r\sigma^2 \left(1 - |\rho|\right) / \left(Z q \hat{q}_d^d\right)} > 0, \quad |\gamma| = \beta \tag{24}$$

with $\text{sign}(\gamma) = -\text{sign}(\rho)$. When $\rho = 0$, HQ are indifferent between $\gamma = \pm\beta$.

When division managers are risk averse, incentive contracts trade off the benefits on uncertainty hedging with the cost of deviating from optimal risk sharing. When division managers face low levels of uncertainty, $\eta \leq \bar{\eta}$, uncertainty aversion does not significantly affect beliefs and, thus, their incentives to exert effort. At these low levels of uncertainty, the disagreement between division managers and HQ is relatively small, with $\hat{q}_d^d = e^{-\eta} q < q$, corresponding to case (iii) in Lemma 1. In this case, the benefits of hedging uncertainty are too small relative to its cost, due to increased risk exposure, and optimal incentive contracts mirror overall those in Theorem 1. The main difference is that the presence of uncertainty, by increasing the cost of incentive provision, reduces both pay-for-performance sensitivity and effort (due to the term $1 - \hat{q}_d^d / q$ in 23). The threshold level $\bar{\eta}(r, \rho)$ is increasing in both division manager risk aversion, r, and correlation coefficient ρ, making this case more relevant when risk-hedging is more valuable.

When division managers are sufficiently exposed to uncertainty on division productivity, $\eta > \bar{\eta}$, HQ find it optimal to hedge their uncertainty and offer incentive contracts with greater cross-division exposure, setting $|\gamma| = \beta > |\rho| \beta$. In this case, the presence of such large uncertainty, if left unchallenged, would significantly depress effort. By granting greater cross-division pay $|\gamma|$, HQ limit pessimism held by division managers, promoting effort, but at the cost of greater risk exposure. To hedge division-manager risk exposure, the sign of the cross-division exposure, γ, is again the opposite to the sign of the correlation coefficient, with $sign(\gamma) = -sign(\rho)$. When the cash-flows of the two divisions are uncorrelated, cross division exposure does not produce any risk-hedging benefit (but only uncertainty hedging), and HQ are again indifferent between setting $\gamma = \pm\beta$.[23]

Optimality of "pure-equity" compensation, $|\gamma| = \beta$, in Theorem 3 is the consequence of division symmetry, leading HQ to grant equal exposure to both two divisions. If divisions are not symmetric, and HQ wishes to implement interior beliefs, as in case (ii) and (iv) of Lemma 1, optimal contracts still involve cross-division exposure, $|\gamma_d| > 0$. However, the composition of pay-for-performance sensitivity, β_d, and cross-division exposure, $|\gamma_d|$, will now depend on the relative size the two divisions (which affects division managers' uncertainty exposure) and their relative risk-exposure.

Corollary 1 *Let the optimal contract be such that both division managers have interior beliefs,*

[23]Interestingly, costly deviations from optimal risk hedging occur only when the benefits from uncertainty hedging are sufficiently large, generating a discrete jump in cross-division exposure, from $|\gamma| = |\rho| \beta$ to $|\gamma| = \beta > |\rho| \beta$. The discontinuity is due to the fact that, with low uncertainty, $\kappa \leq \bar{\kappa}$, division managers beliefs are in case (iii). In this situation, small deviations from optimal risk-sharing have no impact on division managers beliefs, while negatively affecting their welfare. Deviations from optimal risk hedging occur only when they lead to sufficiently large uncertainty-hedging benefits, due to improvements of division managers beliefs, leading HQ to set $|H_d| = 1$.

$|H_d| \in (e^{-\eta_d}, e^{\eta_d})$, and let $a_d q_d > a_{d'} q_{d'}$, for $d \neq d'$. Then the optimal contract $\{\beta_d, \gamma_d\}_{d\in\{A,B\}}$ has

$$\beta_d a_d q_d + r\sigma^2 \beta_d^2 = |\gamma_d| \, a_{d'} q_{d'} + r\sigma^2 \gamma_d^2, \tag{25}$$

with $|\gamma_{d'}| > \xi_{d'}\beta_{d'}$ and $|\gamma_d| < \xi_d \beta_d$.

If the two divisions are of differing size, and the optimal contract induces beliefs that are either in case (ii) or case (iv) of Lemma 1, then the optimal contracts equates the total (expected) cost to HQ of a division manager's exposure to the two divisions, leading to (25). This cost is the sum of two components: for their own division, it is the sum of the (expected) pay-for-performance component, $\beta_d a_d q_d$, and of the corresponding risk premium, $r\sigma^2\beta_d^2$, and for the other division is the sum of cross-pay, $|\gamma_d| \, a_{d'} q_{d'}$, and of the corresponding risk premium, $r\sigma^2\gamma_d^2$. In addition, with respect to the optimal contract in Theorem 2 the presence of risk aversion has the effect increasing cross-division exposure for the relatively smaller division, $|\gamma_{d'}| > \beta_{d'}\xi_{d'}$, and to decrease such exposure for the larger division, $|\gamma_d| < \xi_d\beta_d$.

An important implication of Theorem 3 and Corollary 1 is that optimal incentive contracts have positive cross exposure, $|\gamma| > 0$, even when division managers are risk averse and division cash-flows are not correlated, a clear contrast with the traditional "informativeness principle." This means that the presence of (sufficiently large) uncertainty leads to incentive contracts that would not otherwise be optimal under risk aversion alone.

4 Uncertainty-Averse Principal

Uncertainty aversion by HQ introduces an additional source of disagreement with division managers. We show that the effect of greater disagreement is to increase the cost of relative performance incentive contracts. As a result, when the uncertainty faced by HQ is sufficiently large relative to that faced by division managers, optimal incentive contracts have equity components and no relative performance measures, even in the case of positively correlated division cash-flows. This result is, again, in sharp contrast with the standard optimal contracts absent uncertainty aversion.

Different from uncertainty-neutral principal, beliefs held by uncertainty-averse HQ are not fixed but, rather, are determined endogenously as well. Since the properties of Lemma A1 applies also to HQ, their beliefs $\{\hat{q}_A^{HQ}, \hat{q}_B^{HQ}\}$ are determined by solving

$$\min_{\{\hat{q}_A^{HQ}, \hat{q}_B^{HQ}\} \in K_{HQ}^{\hat{q}}} \pi(\hat{q}^{HQ}) = \sum_{d \in \{A,B\}} E\left[Y_d(a_d) - w_d(Y)\,|\hat{q}^{HQ}\right], \tag{26}$$

where

$$K^{HQ} \equiv \left\{ \hat{q}^{HQ} | \ln\left(\frac{1}{1 - \left|\frac{\hat{q}_A^{HQ}}{q_A} - 1\right|}\right) + \ln\left(\frac{1}{1 - \left|\frac{\hat{q}_B^{HQ}}{q_B} - 1\right|}\right) \leq \eta^{HQ} \right\}. \tag{27}$$

Lemma 3 characterizes HQ beliefs for the case in which HQ have positive residual exposure in either division, $\beta_d + \gamma_{d'} < 1$ (which will be the relevant case in subsequent analysis).

Lemma 3 *Let $\beta_d + \gamma_{d'} < 1$, $d \in \{A, B\}$ with $d' \neq d$, and*

$$H_d^{HQ} \equiv \frac{(1 - \beta_{d'} + \gamma_d) a_{d'} q_{d'}}{(1 - \beta_d + \gamma_{d'}) a_d q_d}, \tag{28}$$

Headquarters assessment of both divisions, $(\hat{q}_A^{HQ}, \hat{q}_B^{HQ})$, is equal to:

i) $\hat{q}_d^{HQ} = q_d$ *and* $\hat{q}_{d'}^{HQ} = e^{-\eta_{HQ}} q_{d'}$ *for* $H_d^{HQ} > e^{\eta_{HQ}}$

ii) $\hat{q}_d^{HQ} = \left[e^{-\eta_{HQ}} H_d^{HQ}\right]^{\frac{1}{2}} q_d,$ *for* $H_d^{HQ} \in \left[e^{-\eta_{HQ}}, e^{\eta_{HQ}}\right]$

iii) $\hat{q}_d^{HQ} = e^{-\eta_{HQ}} q_d$ *and* $\hat{q}_{d'}^{HQ} = q_{d'}$ *for* $H_d^{HQ} < e^{-\eta_{HQ}}$

Similar to Lemma 1, HQ beliefs depend on their relative exposure to the two divisions, as measured by the corresponding uncertainty ratio H_d^{HQ} (note that $H_{d'}^{HQ} = 1/H_d^{HQ}$). When HQ have moderate exposure to both divisions, as in case (ii) with $H_d^{HQ} \in \left[e^{-\eta_{HQ}}, e^{\eta_{HQ}}\right]$, they have conservative beliefs toward each division, $\hat{q}_d^{HQ} < q_d$, and become less confident toward a division when relative exposure to that division increases. When HQ have a sufficiently large exposure to a division, as in cases (i) and (iii) with $H_d^{HQ} > e^{\eta_{HQ}}$ or $H_d^{HQ} < e^{-\eta_{HQ}}$, they will be even less confident toward that division, $\hat{q}_d^{HQ} = e^{-\eta} q_d$, and correspondingly more confident on the other division, $\hat{q}_{d'}^{HQ} = q_{d'}$.

Optimal incentive contracts depend on the extent of uncertainty faced by HQ relative to division managers. We start again with the (simpler) case where division managers are uncertainty averse but risk neutral. Beliefs for division managers are still given in Lemma 1, and their effort levels in Lemma 2. For expositional simplicity, we focus on the case in which division managers are exposed to the same uncertainty: $\eta_A = \eta_B = \eta$.[24] To separate the effect of uncertainty and risk aversion

[24] It is possible, although messy, to extend the analysis to the case in which division managers are exposed to

on optimal incentive contracts, we start again with the case where both HQ and division managers are uncertainty averse but risk neutral.

Theorem 4 *Let both HQ and division managers be uncertainty averse but risk neutral. If divisions are sufficiently homogenous, with* $\hat{H}_d \equiv (Z_{d'}/Z_d)^{1/2} q_{d'}/q_d \in \left(e^{-\eta_{HQ}}, e^{\eta_{HQ}}\right)$ *and the uncertainty faced by HQ is positive but not too large relative to that faced by division managers,* $\eta_{HQ} < \eta - 2\ln\frac{3}{2}$*, optimal incentive contracts have* $H_d^{HQ} = H_d = \hat{H}_d$ *with*

$$\hat{q}_d^d = \hat{q}_d^{d'} = e^{-\frac{\eta-\eta_{HQ}}{2}} \hat{q}_d^{HQ} = e^{-\frac{\eta}{2}} q_d \hat{H}_d^{\frac{1}{2}}, \quad \text{and} \tag{29}$$

$$\hat{q}_d^{HQ} = e^{-\frac{\eta_{HQ}}{2}} q_d \hat{H}_d^{\frac{1}{2}}, \tag{30}$$

for $d, d' \in \{A, B\}$*, and* $d \neq d'$*. Optimal incentive contracts offer pure equity:*

$$\beta_d = \gamma_d = \frac{1}{1 + 3(1 - \hat{q}_d^d/\hat{q}_d^{HQ})} < 1. \tag{31}$$

When divisions are sufficiently homogenous and HQ are not too uncertainty averse relative to division managers,[25] optimal incentive contracts are pure equity, $\beta_d = \gamma_d$. Beliefs, pay-for-performance sensitivity and effort levels mimic those in Theorem 2, with the difference that now HQ beliefs are endogenous and equal $\hat{q}_d^{HQ}$ rather than q_d. Absent risk-aversion, in optimal contracts HQ equate their uncertainty-hedging ratio with respect to each division to the uncertainty hedging ratio of its division manager by setting $H_d^{HQ} = H_d$.

From (31), pay-for-performance sensitivity, β_d, cross-pay, γ_d, and effort level, a_d, now depend on the difference in beliefs held by HQ and division manager, $\hat{q}_d^d/\hat{q}_d^{HQ}$. In turn, from (29) and (30) the difference in beliefs depends of the difference between the uncertainty faced by HQ and division managers, $\eta_{HQ} - \eta < 0$. In particular, an increase of the uncertainty faced by HQ, for given uncertainty faced division managers, increases pay-for-performance sensitivity, cross-pay, and effort. This happens because a smaller difference in uncertainty faced by HQ and division managers reduces the disagreement discount. A smaller discount lowers the cost of incentive provisions and induce HQ to offer contracts with larger pay-for-performance sensitivity, leading to greater effort. Greater pay-for-performance sensitivity, however, increases a division manager's exposure to uncertainty,

different levels of uncertainty, $\eta_A \neq \eta_B$. The optimal contract in Theorem 4 is still equity, $\beta_d = \gamma_d$, but division managers receive different equity shares: $\beta_A \neq \beta_B$.

[25] This condition ensures that HQ has a positive exposure to both divisions, $1 - \beta_d - \gamma_{d'} > 0$, and that their beliefs fall in case (ii) of Lemma 3.

which is offset by a corresponding increase of cross-pay. Beliefs held by HQ and division managers are aligned in the sense that they both hold the same assessment on the relative productivity of both divisions, $\hat{q}_d^{HQ}/\hat{q}_{d'}^{HQ} = \hat{q}_d^d/\hat{q}_{d'}^d$.

Optimal contracts with risk-averse division managers depend again on the trade-off between uncertainty hedging, with its beneficial effect on effort provision, and risk-hedging. With uncertainty averse HQ, they now depend on the level of uncertainty faced by HQ relative to division managers' and on the sign of the correlation coefficient of division cash flows. For tractability, we focus on the symmetric case, condition (S).

Theorem 5 *(Low uncertainty) Let condition (S) hold. There are thresholds* $(\hat{\eta}, \hat{\eta}^{HQ})$ *(defined in the Appendix) such that if* $\eta \leq \hat{\eta}$ *and* $\eta_{HQ} \leq \hat{\eta}^{HQ}$*, optimal incentive contracts induce beliefs for division managers and HQ* $\hat{q}_d^d = e^{-\eta}q < \hat{q}_{d'}^d = q$ *and* $\hat{q}_d^{HQ} = e^{-\frac{\eta_{HQ}}{2}}q < q$ *by setting*

$$\beta = \frac{1}{1 + 2(\rho - \bar{\rho})\left(\frac{\hat{q}_{d'}^d}{\hat{q}_{d'}^{HQ}} - 1\right) + \left(1 - \frac{\hat{q}_d^d}{\hat{q}_d^{HQ}}\right) + \frac{r\sigma^2(1-\rho^2+\bar{\rho}^2)}{Z\hat{q}_d^{HQ}\hat{q}_d^d}}, \quad \gamma = -(\rho - \bar{\rho})\beta, \tag{32}$$

where $\bar{\rho} \equiv \hat{q}_d^d\left(\hat{q}_{d'}^d - \hat{q}_{d'}^{HQ}\right)\frac{Z}{r\sigma^2} = \frac{e^{-\eta}q^2Z}{r\sigma^2}\left(1 - e^{-\frac{\eta_{HQ}}{2}}\right) > 0.$

When overall exposure to uncertainty is sufficiently low, optimal contracts mirror again those absent uncertainty of Theorem 1. The effect of uncertainty is again to reduce pay-for-performance sensitivity, β, and depend on the difference of beliefs between HQ and division managers (captured by the term $\hat{q}_d^d/\hat{q}_d^{HQ}$).

Interestingly, relative-performance compensation, $\gamma < 0$, is now optimal only with sufficiently large correlation, $\rho > \bar{\rho} \geq 0$ (note that $\bar{\rho} = 0$ when $\eta_{HQ} = 0$). The reason is that HQ uncertainty aversion increases the disagreement discount, raising the cost of hedging division manager risk with relative performance compensation. This happens because relative-performance compensation,for division manager d, setting $\gamma < 0$ generates a "short" exposure to the other division, d', while HQ still have a "long" position in that division, $1-\gamma$. From Lemma 3, when HQ are uncertainty averse and hold a long position in d', they are more pessimistic than the reference probability, $\hat{q}_{d'}^{HQ} < q$. In contrast, from Lemma 1, division managers with a short position, $\gamma < 0$, are more confident on the other division d' than the reference probability, $\hat{q}_{d'}^d \geq q$. The combined effect is that HQ and division managers now hold more divergent views on the value of compensation contracts, increasing the disagreement discount and the cost of hedging risk.

The implication is that relative-performance compensation is optimal only when the risk-hedging benefits are sufficiently large, that is, when $\rho > \bar{\rho}$. Correspondingly, the threshold $\bar{\rho}$ is a decreasing function of a division's risk, and of division managers' risk aversion, and is an increasing function of division size (which increases HQ exposure to a division's uncertainty, exacerbating the disagreement discount). If division cash-flows are moderately positively correlated, $0 \leq \rho < \bar{\rho}$, optimal contracts have an equity component, $\gamma > 0$, different from the benchmark case. Finally, HQ and division managers are pessimistic on both divisions, and their assessment of division productivity depends on their relative degree of uncertainty, with $\hat{q}_d^d \gtrless \hat{q}_d^{HQ}$ as $\frac{\eta_{HQ}}{2} \gtrless \eta$.

When uncertainty faced by either HQ or division managers is sufficiently large, optimal incentive contracts depend on the sign of the correlation coefficient between division cash-flows. We start with the case on negatively correlated division cash flows.

Theorem 6 *(Large uncertainty and negatively correlated cash flows) Let condition (S) hold. If $\eta > \hat{\eta}$ or $\eta_{HQ} > \hat{\eta}^{HQ}$ and $\rho \leq 0$ optimal incentive contracts induce beliefs for division managers and HQ equal to $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\eta}{2}}q$ and $\hat{q}_d^{HQ} = e^{-\frac{\eta_{HQ}}{2}}q$ by setting*

$$\beta = \gamma = \hat{\beta} \equiv \frac{1}{1 + 3\left(1 - \hat{q}_d^d / \hat{q}_d^{HQ}\right) + \frac{2r\sigma^2(1+\rho)}{Z\hat{q}_d^{HQ}\hat{q}_d^d}}. \tag{33}$$

When either HQ or division managers cash flows face sufficiently large uncertainty, and division cash-flows are negatively correlated, $\rho \leq 0$, optimal contracts are again pure equity, with $\beta_d = \gamma_d$. Furthermore, in this case, division managers have the same beliefs on the productivity of both divisions, with $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\eta}{2}}q$, for $d, d' \in \{A, B\}$ and $d \neq d'$, and again $\hat{q}_d^d \gtrless \hat{q}_d^{HQ}$ as $\eta_{HQ} \gtrless \eta$. Interestingly, if HQ and division managers face the same uncertainty, $\eta_{HQ} = \eta$, they share the same vision in the firm, $\hat{q}_d^d = \hat{q}_{d'}^d = \hat{q}_d^{HQ} = e^{-\frac{\eta}{2}}q$. Equity-based compensation has the desirable effect of coordinating internal beliefs in the organization, reaching consensus.

With positively correlated cash-flows, optimal incentive contracts depend critically on the degree of uncertainty affecting HQ and division managers.

Theorem 7 *(Positively correlated cash flows) Let condition (S) hold. Let $\rho > 0$, and $\eta > \hat{\eta}$. There are values $(\hat{\eta}_1^{HQ}, \hat{\eta}_2^{HQ})$, with $\hat{\eta}_1^{HQ} < \hat{\eta}_2^{HQ}$, and $\hat{\xi}(\eta_{HQ}) \in (e^{-\eta}, 1)$ (defined in the Appendix) with $\hat{\xi}(0) = 1$, such that* *(i) if* $\eta_{HQ} \leq \hat{\eta}_1^{HQ}$ *optimal incentive contracts induce beliefs equal to* $\hat{q}_d^d = e^{-\frac{\eta}{2}} \hat{\xi}^{\frac{1}{2}} q$ *and* $\hat{q}_{d'}^d > q$*, and* $\hat{q}_d^{HQ} = e^{-\frac{\eta_{HQ}}{2}} q < q$*, by setting*

$$\beta = \frac{1}{1 + \left(\frac{\hat{q}_{d'}^d}{\hat{q}_{d'}^{HQ}} - 1 \right) \hat{\xi} + 2(1 - \frac{\hat{q}_d^d}{\hat{q}_d^{HQ}}) + \frac{2r\sigma^2 \left(1 - \rho\hat{\xi}\right)}{Z \hat{q}_d^{HQ} \hat{q}_d^d}}; \quad \gamma = -\hat{\xi}\beta < 0, \tag{34}$$

where $\hat{\xi}$ *is increasing in* r*,* σ*,* η*, and decreasing in* Z*,* q*, and* η_{HQ}*.*

(ii) if $\eta_{HQ} > \hat{\eta}_2^{HQ}$ *optimal incentive contracts induce beliefs for division managers equal to* $\hat{q}_d^d = \hat{q}_{d'}^d = e^{-\frac{\eta}{2}} q$ *and for HQ equal to* $\hat{q}_d^{HQ} = e^{-\frac{\eta_{HQ}}{2}} q < q$ *by setting* $\beta = \gamma = \hat{\beta}$*.*

When division cash-flows are positively correlated and HQ are exposed to low levels of uncertainty, $\eta_{HQ} \leq \hat{\eta}_1^{HQ}$, while division managers are exposed to large uncertainty, $\eta > \hat{\eta}$, optimal contracts have a relative-performance component, with $\gamma < 0$. Cross-division exposure is again proportional to pay-for-performance sensitivity by a factor $\hat{\xi}$, which represent the hedging component of division manager compensation. Importantly, the hedging factor $\hat{\xi}$ depends now on the level of division managers' risk aversion and their exposure to uncertainty, relative to the uncertainty faced by HQ. Greater managerial risk aversion and cash-flow risk increase the importance of hedging division manager's risk, leading to more cross-division exposure (bigger $\hat{\xi}$). Similarly, greater uncertainty aversion by division managers increases the importance of uncertainty hedging, leading again to more cross-division exposure. In contrast, greater uncertainty by HQ (greater η_{HQ}) and exposure to division uncertainty (larger values of Z and q), by exacerbating the disagreement discount, increase the cost of both risk and uncertainty hedging. The effect is to reduce optimal cross-division exposure, γ, worsening division managers' confidence in their own division: $\hat{q}_d^d = e^{-\frac{\eta}{2}} \hat{\xi}^{\frac{1}{2}} q$ (where $\hat{\xi} < 1$).

When HQ are exposed to sufficiently large uncertainty, $\eta_{HQ} > \hat{\eta}_2^{HQ}$, optimal incentive contracts are again pure equity with $\beta = \gamma$, with no relative-performance compensation even when division cash-flows are positively correlated. The reason is that large uncertainty exacerbates disagreement on relative-performance compensation and results into a more significant cost of hedging division-manager risk. In this situation, hedging risk can conflict with hedging uncertainty. With sufficiently large uncertainty, the uncertainty-hedging motive overcomes the risk-hedging motive, and HQ forego altogether the risk-hedging benefits of relative-performance. Rather, they offer pure-equity contracts that better aligns division managers beliefs with theirs, lowering the cost of incentive provision and promoting effort. This case is an important reversal of the predictions of the standard optimal contracting problem with no uncertainty of Theorem 1.

Finally, note that equity compensation when HQ are uncertainty averse is optimal even in the case of heterogenous divisions.

Corollary 2 *Let the optimal contract be such that HQ granting positive exposure to both divisions,* $\beta_d, \gamma_d > 0$*, and both division managers, as well as HQ have beliefs as in case (ii) of Lemma (1) and (3), with* $H_d \in (e^{-\eta_d}, e^{\eta_d})$ *and* $H_d^{HQ} \in \left(e^{-\eta_{HQ}}, e^{\eta_{HQ}}\right)$*. Then the optimal contract* $\{\beta_d, \gamma_d\}_{d\in\{A,B\}}$ *has*

$$\beta_d a_d \hat{q}_d^{HQ} + r\sigma^2\beta_d^2 = \gamma_d a_{d'} \hat{q}_{d'}^{HQ} + r\sigma^2\gamma_d^2. \tag{35}$$

In addition, $(1-\beta_d-\gamma_{d'})a_d\hat{q}_d^{HQ} = (1-\beta_{d'}-\gamma_d)a_{d'}\hat{q}_{d'}^{HQ}$*, and HQ grants equity compensation:* $\beta_d = \gamma_d$*.*

Similar to Corollary (1), the optimal contract with interior beliefs for both HQ and division managers equates the total (expected) cost to HQ of a division manager's exposure to both division, giving (35). Different from Corollary 1, however, $\hat{q}^{HQ}$ is now endogenous. From Lemma 3, when HQ has interior beliefs, HQ equate expected exposure to each division, $(1-\beta_d-\gamma_{d'})a_d\hat{q}_d^{HQ} = (1-\beta_{d'}-\gamma_d)a_{d'}\hat{q}_{d'}^{HQ}$, which implies that $\beta_d = \gamma_d$. Corollary 2 shows that, when HQ are uncertainty averse, optimality of equity compensation is the outcome of HQ desire to align division managers beliefs with theirs.

5 Robustness and Discussion

An important assumption in our paper is that the core belief set is a strictly convex set with smooth boundaries, a property satisfied by the relative entropy criterion. Strict convexity guarantees that belief held by uncertainty averse division managers and HQ respond to changes in compensation contracts. This property does not hold when agents hold "rectangular" beliefs, for example where the core belief set is

$$K^i(q^i) \equiv \{\hat{q}^i : [q_A - \eta^i \le \hat{q}_A^i \le q_A + \eta^i] \times [q_B - \eta^i \le \hat{q}_B^i \le q_B + \eta^i].$$

With rectangular beliefs, uncertainty averse agents do not benefit from uncertainty hedging. Intuitively, the solutions to the minimization problems in the RHS and LHS of (2) are equivalent, and the condition holds as an equality. In the context of our paper, division managers and HQ beliefs on division productivity, the solutions to (14) and (26), are determined by a fixed "worst case scenario" and do not depend on the relative exposure to division uncertainty generated by incentive contracts.

Our prediction on the impact of uncertainty aversion on inclusion of relative performance in incentive contract discussed in Section 4 hold also in the extreme case of "rectangular" beliefs. This happens because relative performance contracts, where division managers and HQ have opposite exposure to cross-division cash-flow, lead them to hold extreme opposite beliefs on division productivity. Specifically, in this case HQ hold a long position in both divisions and set beliefs at the lower extreme of the belief range, $q_i^{HQ} - \eta^{HQ}$. In contrast, division managers hold a short position in the cross-division cash-flow, and are concerned when that division has high productivity, and will set beliefs at the higher extreme of the belief range, $q_{d'}^d + \eta^d$. Thus, rectangular beliefs lead to extreme disagreement between HQ and division managers on the value of incentive contracts with relative performance, exacerbating the disagreement discount. The effect is to make even more costly, from the point of view of HQ, to meet division managers' participation constraint. As discussed in Section 4, increasing the cost of hedging division managers' risk through relative performance makes such contracts less desirable. [26]

One of the main results of our paper is to establish the benefit of uncertainty hedging for incentive provision. We study the case where HQ can hedge division manager uncertainty by offering cross-pay, either as equity-based or relative-performance compensation. More generally, HQ can hedge division manager uncertainty by making incentive contracts to depend also on other external variables, such as an appropriate benchmark. The potential benefit of inclusion of benchmarks in incentive pay to hedge uncertainty, however, must be balanced against two costs. The first cost is due to the additional risk exposure that it may impose on division managers. The second cost is that external benchmarks may represent side bets for both HQ and division managers. In this case, HQ and division managers may hold opposite positions on the benchmark, exacerbating the

[26] Indeed, when there is sufficient uncertainty, $\eta^d + \eta^{HQ}$, the optimal contract is $\gamma_d = 0$ and $\beta_d = \frac{1}{2 - \frac{\hat{q}_A^A}{\hat{q}_A^{HQ}} + \frac{r\sigma^2}{Z\hat{q}_A^{HQ}\hat{q}_A^A}}$.

disagreement discount, making it more for HQ to meet division managers' participation constraint. Our paper suggests that inclusion of external benchmarks in compensation contracts may help hedge division manager uncertainty, with a beneficial effect on the incentives to exert effort. Such benefits, however, must be balanced against an increase in the cost of incentive provision due to greater risk exposure and disagreement discount.

6 Uncertainty and Beliefs in Organizations

We develop a novel theory of belief formation in organizations based on uncertainty aversion. We argue the presence of uncertainty, and aversion to it, can generate belief heterogeneity even in cases where agents share the same set of "core beliefs." Belief heterogeneity emerges endogenously as the consequence of agents' differential exposure to the sources of uncertainty.

Individual exposure to uncertainty can be determined first by their position in the organization. Top executives are exposed to all the uncertainty factors that affect a firm, either directly, or through the relevant economic environment surrounding their firm. In contrast, division managers are disproportionally exposed to uncertainty factors affecting their own division. Exposure to division uncertainty may derive, for example, from the impact of firm performance on division managers' human capital, affecting career opportunities within the firm or their outside options. We refer to this exposure to uncertainty as hierarchical exposure, because it depends on an agent's position in the organizational hierarchy.

The second form of exposure depends on the contractual arrangements in the organization. Division managers make choices in the context of a web of contracts and rules (organizational protocols) that govern firms. We refer to this exposure to uncertainty as contractual exposure, because it depends on all contractual arrangements surrounding agents.

The structure of beliefs that emerges in equilibrium is endogenous and depends on both its hierarchical configuration and the contractual relationships that bind agents together. An implication of our paper is that internal beliefs can be managed by both organization design and contract design. In this paper, we focus on the latter. We argue that, by proper design of incentive contracts, HQ can affect beliefs within the organization and induce a more favorable belief system, promoting efficiency.

We show that disagreement emerges as an equilibrium outcome in the belief structure in an organization. For example, managers in the upper levels of the hierarchy can (endogenously) be more confident about their firm's future performance than lower-level employees. This implies that rank-and-file managers perceive members of the top management team of a firm (such as CEOs and CFOs) as overconfident and unrealistically optimistic.

The extent of internal disagreement depends on the level of uncertainty that characterize different layers in the organization. When the upper levels in the hierarchy are relatively less concerned about uncertainty than lower-levels, uncertainty concerns deeper down in the hierarchy can generate significant disagreement in the organization. HQ can respond by designing contracts with greater cross-division exposure, through either a more significant equity-based compensation (when division cash-flows are positively correlated) or enhanced relative-performance provisions (with negatively correlated cash-flows).

Our model provides a theoretical foundation of the links between compensation structure and beliefs systems in organizations.[27] The effect of equity-based compensation is to realign internal beliefs, promoting a shared view and internal consensus. In contrast, relative-performance compensation has two divergent effects on internal beliefs. First, it improves and realigns a division managers' beliefs on their division with those of HQ, with beneficial effect on effort provision. The disadvantage of relative-performance compensation is that it may lead division managers to be more confident on the other divisions in the firm, relative to theirs, creating envy and discord. Such discord may interfere with overall management and performance of the organization, for example by affecting the internal allocation of resources.

Finally, a large exposure to uncertainty by top levels in the organization increases the cost of relative-performance compensation. In this situation, HQ may prefer to forego the risk-hedging benefits of relative-performance and, rather, offer cheaper equity-based contracts. Such equity-based contracts provide uncertainty-hedging and promote effort, with the additional benefit of fostering consensus.

7 Empirical Implications

Our paper has several empirical implications that can help explaining some otherwise puzzling features of the compensation policies adopted by corporations.

[27] Links between pay and sentiment is shown in several papers, such as Bergman and Jenter (2007), Heaton (2002), and Oyer and Schaefer (2005), among others.

1. *Firms characterized by high uncertainty, such as young firms, prefer compensation contracts with an equity component rather than relative performance.* A puzzling feature of the compensation structure of many young firms is the widespread use of equity-based compensation throughout the organization. While equity-based compensation appears to be justified for members of the top management, such as the CEO, it is less clear why lower-level managers should receive equity-based compensation. This is because equity-based compensation reduces the sensitivity of managerial pay to their action, and thus reduces its effectiveness as an incentive. This practice is even less justifiable for low-level employees, where the connection between an employee's actions and equity value is even more tenuous.

Our paper provides an explanation for the common occurrence of equity-based compensation and the infrequent use of relative-performance assessments. Specifically, equity-based compensation plays two important roles. First, it better aligns the beliefs of members of the organization with the one held by the top management. Absent the equity component in pay, individuals would hold more conservative beliefs than the top management on the expected performance of their unit. Inclusion of the equity-based compensation aligns their expectations with the ones held by the top management, improving the overall disposition of the organization. The second benefit is that, because of the improvement of expectations, employees will exert greater effort, improving firm value.

2. Relative performance and pay-for-luck. It is often suggested that lack of relative-performance component in executive pay results in rewarding top managers for performance influenced by market forces outside their control rather than their own efforts ("pay-for-luck").[28] Our paper suggests an advantage of equity-based compensation over relative-performance. Relative-performance creates a divergence between shareholders, who typically hold "long" positions in their portfolios, and top executives, that would hold "short" positions in the benchmarks adopted as a basis for their relative performance. The presence of such divergent positions has the consequence of creating potentially harmful disagreement between shareholders and top management. Equity-based compensation, in contrast, has the benefit of aligning shareholders and top executives exposure to uncertainty, preserving agreement.

3. Mature firms adopt compensation contracts primarily based on pay-for-performance measures

[28] Gopalan, Milbourn and Song (2010) argue this is a response to strategic uncertainty surrounding firms.

with relative-performance components. As firms mature, the level of uncertainty surrounding their business activities decreases, reducing (or even eliminating) the need for equity-based compensation. For these firms, effort levels in the organization is better elicited by the use of pay-for-performance incentive contract, making equity-based compensation redundant. This means that firms should first start, when they are young, with incentive contracts heavily skewed toward equity-based compensation, and then move toward pay-for-performance based contracts as they mature.

4. Optimal compensation in business groups. Our paper has also implications for the compensation structure in business groups. Consider an executive manager in a subsidiary of a business (or family group). Traditional theory would suggest that in these cases compensation should depend only on the performance of their subsidiary or business unit. In contrast, compensation for such managers is often tied to the performance of the entire business group. For example, Ma, Tang, and Gomez (2019) study the compensation structure for the mutual funds industry and find that in about half of their sample, managers' bonuses are directly linked to the overall profitability of the advisor. A similar practice is common in the investment bank industry, where individual bonuses depend also on the overall performance of the intermediary. Such features, which would be difficult to be justified on the basis of risk-aversion only, are consistent with the findings of our paper.

5. Managerial (over)optimism. Our model predicts that managers in the upper echelon of corporate ladders tend to be relatively more optimistic about their firm's future performance. This implies that, rank-and-file managers perceive members of the top management team of a firm (such as CEOs and CFOs) as overconfident and unrealistically optimistic. The role of managerial overconfidence in corporations has been extensively documented (see, for example, Heaton, 2002, and Malmendier and Tate, 2005, among others). Goel and Thakor (2008) suggest that managerial optimism can be the outcome of the managerial selection process, whereby lucky and overconfident managers are more likely to rise to the top positions of companies. Our paper suggests that top managers' optimism is the consequence of uncertainty hedging, and not necessarily the sign of a negative behavioral bias.

6. Entrepreneur CEOs and family wealth. Entrepreneurship is commonly associated with family wealth (Hurst and Lusardi, 2004), and access to family wealth is a primary determinant of entrepreneurship (Levine and Rubinstein, 2017). There are several reasons why family wealth may be associated with greater incentives to become entrepreneurs and, thus, CEOs. These include relaxation of financial constraints and greater diversification opportunities (lower cost of capital). Note that traditional risk-diversification rationales would imply the wealthy families invest in industries with low (or negative) correlation with the bulk of family money. Our paper adds a novel rationale for the association between family wealth and entrepreneurship. Individuals in wealthy families, by virtue of their broad portfolio, benefit more from uncertainty hedging, giving them a comparative advantage in investing in business surrounded by greater uncertainty. As a consequence, owners/CEOs belonging to wealthy families would (endogenously) be characterized by more optimistic views of their companies. These are new and testable implications.

8 Conclusions and Future Research

We examine the impact of uncertainty aversion on the design of optimal incentive contracts in an organization. We study the problem faced by a multidivisional firm, for simplicity with two divisions, where agents may be uncertainty averse. Divisional managers exert unobservable effort that affects the productivity of their division, creating moral hazard. The contracting problem is complicated by the fact that division managers are uncertainty averse, making them unduly conservative in the eyes of their HQ. Such disagreement is endogenous, and the outcome of risk-exposure created in incentive contracts to promote effort.

We showed that the structure of optimal incentive depends on the level of uncertainty that affects firms. For firms with low uncertainty, incentive contracts still exhibit pay-for-performance compensation when division cash-flows are negatively correlated, and relative-performance compensation when division cash-flows are positively correlated, but less than the no-uncertainty case. For firms characterized by high levels of uncertainty, optimal incentive contracts are more likely to have cross-pay compensation or straight-equity.

The analysis in our paper can be extended in several ways. First, it would be interesting to examine multi-tasking situations, as in Holmström and Milgrom (1991). Our paper suggests an important aspect of uncertainty hedging and its impact on task assignment and optimal compensation. An additional avenue of research is to determine the impact of uncertainty on organization design: it is plausible to expect that organizations in highly uncertain environments have a relatively flat structure, to promote uncertainty hedging. Our paper is also essentially a partial equilibrium model. An interesting question is to examine the impact of labor market forces in a process where heterogenous agents are matched with heterogenous firms. We leave these important questions for future research.

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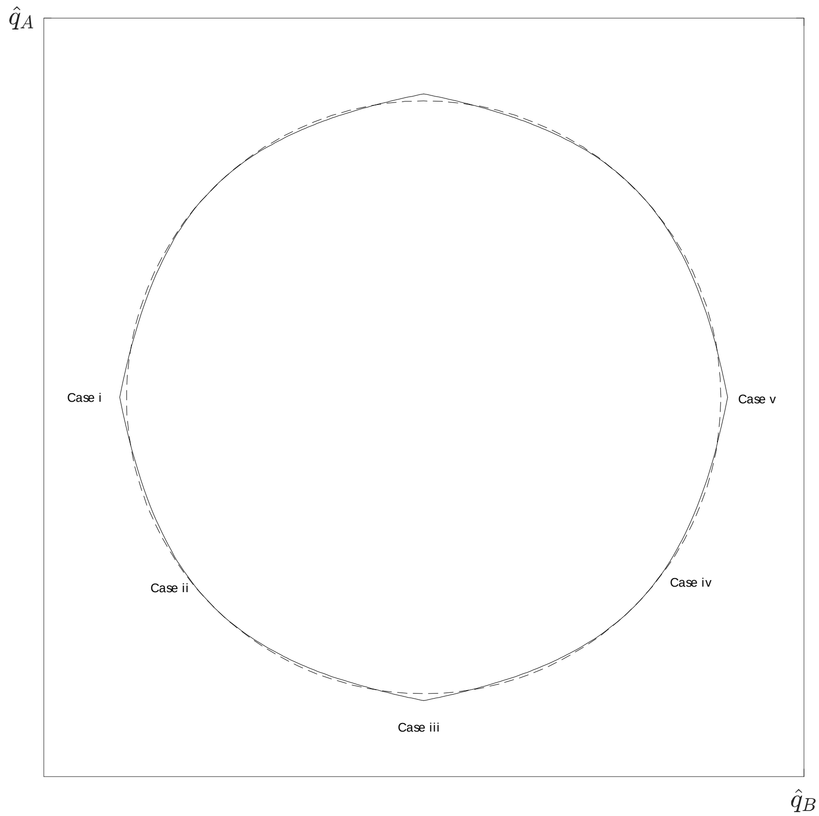

Figure 1: Core of Beliefs

The figure displays the core-belief set, Equation (A35), and the 5 cases of Lemma A1 for $d = A$ under parameter values $q_A = q_B = 100$ and $\eta_A = \ln(5)$. In Case (i), $H_A > e^{\eta_A}$, and the division manager holds the reference beliefs toward her own division, $\hat{q}_A = q_A$, and extreme pessimism toward the other division, $\hat{q}_B << q_B$. In Case (ii), $H_A \in (e^{-\eta_A}, e^{\eta_A})$, leads to moderate pessimism toward both divisions, $\hat{q}_d < q_d$, $d \in \{A, B\}$. In Case (iii), $H_A \in (-e^{-\eta_A}, e^{\eta_A})$, leads to extreme pessimism toward her own division, $\hat{q}_A << q_A$, and to reference beliefs toward the other division, $\hat{q}_B = q_B$. In Case (iv), $H_A \in (-e^{\eta_A}, -e^{-\eta_A})$, leads to moderate pessimism toward her division, $\hat{q}_A < q_A$, and to optimism toward the other division, $\hat{q}_B > q_B$. In Case (v), $H_A < -e^{\eta_A}$, leads again to hold the reference beliefs toward her own division, $\hat{q}_A = q_A$, and to be very confident toward the other division, $\hat{q}_B >> q_B$. The dotted line represents the core of beliefs from Equation (3.12) of Chen and Epstein (2002), with $(\hat{q}_A - q_A)^2 + (\hat{q}_B - q_B)^2 \leq k_A$: this set corresponds to the relative entropy criterion for symmetric effort and zero correlation.

A Internet Appendix A: Aggregation and Linearity

In this appendix we study a continuous-time dynamic stochastic model, and we show that the aggregation and linearity results of Holmström and Milgrom (1987) hold in our environment with uncertainty. The basic model is modified as follows. Time is continuous, $t \in [0,1]$, and at each instant each division manager chooses effort, $a_{d,t} \in \mathbb{R}_+$, affecting the probability distribution of divisional cash-flows. We assume that cash-flows of both divisions, $Y_t \equiv (Y_{A,t}, Y_{B,t})$, follow the (joint) process

$$dY_t = \mu_t dt + \Gamma dW_t, \tag{A1}$$

where $W_t = (W_{A,t}, W_{B,t}) \in \mathbb{R}^2$ is a standard bivariate Brownian motion defined on the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P^a\right)$, with $Y_{A,0} = Y_{B,0} = 0$. Note (Y, W, P^a) is a weak solution to the stochastic differential equations in (A1); all processes are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Following Holmström and Milgrom (1987), we assume that division manager efforts affect only the drift of its own division with no externalities (or synergies) across divisions. The marginal product of a division manager's effort is greater in more productive divisions, and we set $\mu_t \equiv (\mu_{A,t}, \mu_{B,t})'$ with $\mu_{d,t} = a_{d,t} q_d$, where q_d represents the productivity of division $d \in \{A, B\}$ under the probability, P^a. We will refer to division managers' (joint) action profile as $a_t = (a_{A,t}, a_{B,t})'$. Division cash-flows are homoskedastic, with constant variance σ^2, and may be (positively or negatively) correlated, with correlation coefficient ρ. Further, we assume effort does not affect the variance-covariance matrix, Σ.[1] Thus, Γ is the symmetric square root of the variance-covariance matrix, giving

$$\Gamma \equiv \begin{bmatrix} \frac{\sigma}{2}\left(\sqrt{1+\rho}+\sqrt{1-\rho}\right) & \frac{\sigma}{2}\left(\sqrt{1+\rho}-\sqrt{1-\rho}\right) \\ \frac{\sigma}{2}\left(\sqrt{1+\rho}-\sqrt{1-\rho}\right) & \frac{\sigma}{2}\left(\sqrt{1+\rho}+\sqrt{1-\rho}\right) \end{bmatrix}, \ \Sigma = \Gamma'\Gamma = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}. \tag{A2}$$

Exerting effort is costly: each division manager suffers an instantaneous monetary cost $c_d(a_{d,t})\, dt$, where $c_d : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable, increasing and convex function. While in the body of the paper, we set $c_d(a_{d,t}) = \frac{1}{2Z_d} a_{d,t}^2$, where Z_d characterizes effort efficiency of division managers, our result in this appendix applies generally for any increasing and convex cost function. Following Holmström and Milgrom (1987), division managers and HQ exhibit preferences with constant absolute risk aversion (CARA), and are paid and consume only at the end of the period, $t = 1$.[2]

Effort exerted by each division manager is not observable by either HQ or the other division manager, creating moral hazard. HQ promote effort by offering division managers incentive contracts, $\{w_d\}_{d\in\{A,B\}}$, as follows. We assume that output from each division, $Y_{d,t}$, is publicly observable, and we let $h_t = \{Y_s | s \leq t\}$ represent the entire history of cash-flows from both divisions at each point in time t. HQ can condition compensation to each division manager on the entire history, that is $w_d(h_1)$. We impose the usual square-integrable condition that $E^{P^a}[w_d(h_1)]^2 < \infty$. Given an incentive contract $w_d(h_1)$ and effort level process $a_d \equiv \{a_{d,t}\}_{t\in[0,1]}$, division manager $d \in \{A, B\}$ earns an end-of-period payoff

$$U_d(h_1) \equiv u\left(w_d(h_1) - \int_0^1 c_d(a_{d,t})\, dt\right), \tag{A3}$$

where $u(w) = -e^{-rw}$, and r is the coefficient of absolute risk aversion for both divisional managers. Similarly, HQ earn end-of-period payoff equal to

$$\Pi(h_1) \equiv \pi\left(Y_{A,1} + Y_{B,1} - w_A(h_1) - w_B(h_1)\right), \tag{A4}$$

where $\pi(X) = -e^{-RX}$, and R is the coefficient of absolute risk aversion for HQ. Because processes are in L^2, they both have finite expectation.

The differential game unfolds as follows. At the beginning of the period, $t = 0$, HQ choose incentive contracts $w_d(h_1)$ for each division manager $d \in \{A, B\}$. HQ can commit to contracts $\{w_d(h_1)\}_{d\in\{A,B\}}$, which are observable to both managers. After incentive contracts are offered and accepted, division managers continuously and simultaneously

[1] Hemmer (2017) and Ball et al. (2020) study contracts when effort affects Σ.

[2] Restricting pay and consumption only to the end avoids the complication of intertemporal consumption smoothing via private savings. This is studied in He et al. (2017), who study a dynamic agency problem in a setting without Knightian uncertainty.

choose their level of effort, $a_{d,t}$, after observing history h_t. At the end, $t = 1$, division managers are compensated according to the realized history, h_1, and consumption takes place.

We model uncertainty aversion by adopting the minimum expected utility (MEU) approach of Chen and Epstein (2002), a dynamic extension of Gilboa and Schmeidler (1989). We assume that both HQ and division managers are treat the probability measure P^a as uncertain. Following Chen and Epstein (2002), P^a represents a "reference probability," which is assumed to be common for both division managers and HQ.[3] In addition, we consider beliefs distortions that are mutually absolutely continuous measures with respect to P^a, allowing us to use Girsanov's Theorem.[4] Define a density generator to be a $\mathbb{R}^2$-valued $\mathcal{F}_t$-predictable process θ_t satisfying the Novikov condition

$$E^{P^a}\left[\exp\left(\frac{1}{2}\int_0^1 \theta_s \cdot \theta_s ds\right)\right] < \infty, \tag{A5}$$

so that the process

$$z_t^\theta \equiv \exp\left\{-\frac{1}{2}\int_0^t \theta_s \cdot \theta_s ds - \int_0^t \theta_s dW_s\right\} \tag{A6}$$

is a $(P^a, \mathcal{F}_t)$ martingale. By Girsanov's Theorem, the process θ_t generates an equivalent probability measure $\tilde{P}^{a,\theta}$ on $(\Omega, \mathcal{F})$ such that

$$\frac{d\tilde{P}^{a,\theta}}{dP^a}|_{\mathcal{F}_t} = z_t^\theta, \tag{A7}$$

where z_t^θ is the Radon-Nikodym derivative of $\tilde{P}^{a,\theta}$ with respect to P^a when restricted to $\mathcal{F}_t$. Note that, from Girsanov's Theorem, the process

$$W_t^\theta = W_t + \int_0^t \theta_s ds, \tag{A8}$$

is a standard Brownian motion under the new measure $\tilde{P}^{a,\theta}$.

Under the measure $\tilde{P}^{a,\theta}$, divisional cash-flows Y^θ follow the process

$$dY_t^\theta = Qa_t dt + \Gamma\left(dW_t^\theta - \theta_t dt\right) = \mu^\theta(a_t)dt + \Gamma dW_t^\theta, \tag{A9}$$

where

$$Q \equiv \begin{bmatrix} q_A & 0 \\ 0 & q_B \end{bmatrix} \quad \text{and} \quad \mu^\theta(a_t) \equiv Qa_t - \Gamma\theta_t. \tag{A10}$$

Thus, the density generator process θ_t describes decision makers' ("distorted") beliefs on the instantaneous productivity of both divisions.[5]

Following Chen and Epstein (2002), we assume that uncertainty is IID, and we allow for the possibility that HQ and division managers may be exposed to different degrees of uncertainty. Thus, we let $\theta_{\delta,t} \in K_\delta(a_t)$, $\delta \in \{HQ, A, B\}$, for all $t \in [0,1]$, where $K_\delta \in \mathbb{R}^2$ is set-continuous (both upper- and lower-hemicontinous) with $K_\delta(a_t)$ a convex set for all a_t. We allow for the possibility that uncertainty depends of the level of effort, a_t. Let

$$\mathcal{P}_\delta^\Theta(a_t) = \left\{\tilde{P}^{a,\theta} | \theta_t \in K_\delta(a_t), \forall t\right\} \tag{A11}$$

be the set of admissible priors for division managers and HQ. Note that $\tilde{P} \in P_\delta^\Theta(a_t)$ if and only if there is a $\theta_{\delta,t}$ such that $\tilde{P} = \tilde{P}^{a,\theta}$ and $\theta_{\delta,t} \in K_\delta(a_t)$ for all t. IID uncertainty can be interpreted as nature drawing independent increments $dW_t^{a,\theta}$ of the process $W_t^{a,\theta}$ from different urns at each point in time. Similar to Chen and Epstein (2002), these assumptions imply that a division's past cash-flow realizations are not informative on future cash-flows, thus excluding learning. Importantly, they ensure that divisional managers and HQ face stationary uncertainty. Note that the core beliefs set is rectangular over time, as required for time consistency by Chen and Epstein (2002). However, the set K may not be a rectangle. Indeed, similar to Chen and Epstein (2002), Equation 3.12, we will consider strictly convex ("round") sets $K_\delta(a)$, $\delta \in \{HQ, A, B\}$ with smooth boundaries.

At the beginning of the game, $t = 0$, HQ offer division managers a pair of contracts, $w(h_1) \equiv \{w_d(h_1)\}_{d\in\{A,B\}}$,

[3] Hansen et al. (2006) refer to the measure P^a as the "approximating model."

[4] Miao and Rivera (2016) and Szydlowski and Yoon (2022) use a similar approach.

[5] Note that in the general model, we allow for the possibility that a division manager considers more uncertain the productivity of the other division with respect their own.

and a set of (history-dependent) instructions $a \equiv \{a_d\}_{d\in\{A,B\}}$ to maximize expected payoff, that is to solve

$$\max_{\{w,a\}} \min_{\tilde{P}\in\mathcal{P}^{\Theta}_{HQ}(\{a_d,a_{d'}\})} E^{\tilde{P}}\pi\left(Y_{A,1}+Y_{B,1}-w_A\left(h_1\right)-w_B\left(h_1\right)\right) \tag{A12}$$

subject to the constraints that (i) each division managers choose an effort process, a_d, given the other division manager's action profile, to solve

$$\max_{\tilde{a}_d} \min_{\tilde{P}\in\mathcal{P}^{\Theta}_{d}(\{\tilde{a}_d,a_{d'}\})} E^{\tilde{P}}_t u\left(w_d\left(h_1\right)-\int_0^1 c_d\left(\tilde{a}_{d,t}\right)dt\right), \tag{A13}$$

and (ii) the pairs $\{a_d, w_d(h_1)\}_{d\in\{A,B\}}$ satisfy their participation constraints

$$\min_{\tilde{P}\in\mathcal{P}^{\Theta}_{d}(\{a_d,a_{d'}\})} E^{\tilde{P}}_0 u\left(w_d\left(h_1\right)-\int_0^1 c_d\left(a_{d,t}\right)dt\right)\geq u_0=0 \tag{A14}$$

for $d,d'\in\{A,B\}$, and $d\neq d'$, where u_0 is a division manager's reservation utility, normalized to zero. Note that in problem (A12) - (A14) a division manager's uncertainty exposure is endogenous and is determined by the incentive contract, $w_d(h_1)$, offered by HQ. Contractual exposure concurs to determine a division manager's beliefs, $\tilde{P}_d$. Given their higher-level position in the firm hierarchy, HQ exposure to uncertainty is determined by their residual claim in firm cash-flow, given incentive contracts offered to both division managers in the firm.[6] HQ hierarchical exposure concurs to determine HQ beliefs, $\tilde{P}_{HQ}$. The triplet $\{\tilde{P}_{HQ},\tilde{P}_A,\tilde{P}_B\}$ determines the belief structure prevalent in the firm.

Definition A1 *An equilibrium is a set of contracts , $w(h_1)\equiv\{w_d(h_1)\}_{d\in\{A,B\}}$, and action processes $\{a_A,a_B\}$, such that:*

- *(i) Given incentive contracts $w(h_1)$, for every history h_t each division manager selects effort, a_d, optimally, solving (A13), given the other division manager's action process, $a_{d'}$ for $d'\neq d$;*
- *(ii) HQ offer contracts $w(h_1)$ that maximizes (A12) subject to (A13) - (A14).*

The aggregation and linearity property of Holmström and Milgrom (1987) holds in the case of stationary (IID) uncertainty with two division managers.

Theorem A1 *The optimal contract between HQ and division managers is linear in cash-flows, $w_d(h_1)=s_d+\beta_d Y_{d,1}+\gamma_d Y_{d',1}$, with constant s_d, β_d, γ_d, for $d,d'\in\{A,B\}$, and $d\neq d'$; it induces constant effort, $a_{d,t}=a_d$, and beliefs, $\tilde{P}^{a,\theta}$, with constant distortions, $\theta_{d,t}=\theta_d$ and $\theta_{HQ,t}=\theta_{HQ}$, for all t.*

Proof of Theorem A1. Each division manager selects a_t to maximize

$$U_{d,t}\equiv \min_{\tilde{P}\in\mathcal{P}^{\Theta}_{d}(\{\tilde{a}_d,a_{d'}\})} E^{\tilde{P}}_t u\left(w_d\left(h_1\right)-\int_0^1 c_d\left(\tilde{a}_{d,t}\right)dt\right). \tag{A15}$$

Given optimal effort a^* and worst-case scenario process, θ^*_d, by the martingale representation theorem, $U_{d,t}$ is an Itô Process adapted to Y^{a^*,θ^*} with zero drift (Theorem 4.33, Jacod and Shiryaev, 1987): $dU_{d,t}=\Upsilon'_{d,t}\left[dY^{a^*,\theta}_t-\mu^{\theta}\left(a^*\right)dt\right]$. Define $w_{d,t}$ so that $u(w_{d,t}-\int_0^1 c_d\left(a^*_{d,t}\right)dt)=U_{d,t}$: $w_{d,t}$ can be interpreted as the balance that the DM has with the HQ. We can express

$$w_{d,t}=\int_0^1 c_d\left(a^*_{d,t}\right)dt-\frac{1}{r}\ln\left(-U_{d,t}\right). \tag{A16}$$

Note that this is the equilibrium a^*, not necessarily the chosen a. Let $\phi=\int_0^1 c_d\left(a^*_{d,t}\right)dt-\frac{1}{r}\ln\left(-U\right)$. Because $\frac{\partial\phi}{\partial t}=0$, $\frac{\partial\phi}{\partial U}=\frac{1}{(-rU)}$, and $\frac{\partial^2\phi}{\partial U^2}=\frac{1}{rU^2}$, and defining $B_{dt}\equiv\frac{1}{(-rU)}\Upsilon_{dt}$,

$$dw_{d,t}=B'_{d,t}\left(dY^{a^*,\theta}_t-\mu^{\theta}\left(a^*_t\right)dt\right)+\frac{r}{2}B'_{d,t}\Sigma B_{d,t}dt \tag{A17}$$

[6] We assume HQ are full residual claimants in firm cash-flow. More generally, HQ act in the context of incentive contracts from a compensation committee, exposing them to contractual exposure as well.

Because the optimal contract is progressively measurable with respect to $\mathcal{F}_t^Y$, we can express

$$w_{d,t} = w_{d,0} + \int_0^1 c_d\left(a_{d,t}^*\right)dt + \int_0^t B_{d,t}^{'}\left(dY_t^\theta - \mu^\theta\left(a_t^*\right)dt\right) + \int_0^t \frac{r}{2}B_{d,t}^{\prime}\Sigma B_{d,t}dt \tag{A18}$$

At the optimal effort level, a_t^*, this becomes

$$w_{d,t} = w_{d,0} + \int_0^1 c_d\left(a_{d,t}^*\right)dt + \int_0^t B_{d,t}^{\prime}dW_t^{a^*,\theta} + \int_0^t \frac{r}{2}B_{d,t}^{\prime}\Sigma B_{d,t}dt \tag{A19}$$

Off-equilibrium, if the DM deviates from a^* to a, he shifts the probability distribution from P^{a^*} to P^a. From the Girsanov's Theorem, set $h^a = \Gamma^{-1}\Delta\mu$, where $\Delta\mu$ is a vector with $\Delta\mu_d = \mu_d\left(a_d\right) - \mu_d\left(a_d^*\right)$ and $\Delta\mu_{d'} = 0$. Let $z_t^a \equiv \exp\left\{-\frac{1}{2}\int_0^t h_s^a \cdot h_s^a ds + \int_0^t h_s^a dW_s\right\}$. By Girsanov's Theorem, h^a generates an equivalent probability measure P^a on $(\Omega, \mathcal{F})$ such that $\frac{dP^a}{dP^{a^*}}|_{\mathcal{F}_t} = h^a$. Further, $W_t^a = W_t^{a^*} - \int_0^t h_s^a ds$ is a Brownian Motion. This changes the payoff to

$$w_{d,t} = w_{d,0} + \int_0^1 c_d\left(a_{d,t}^*\right)dt + \int_0^t B_{d,t}^{'}\left(\mu^\theta\left(a_t\right) - \mu^\theta\left(a_t^*\right)dt\right) + \int_0^t \frac{r}{2}B_{d,t}^{\prime}\Sigma B_{d,t}dt + \int_0^t B_{d,t}^{\prime}dW_t^a \tag{A20}$$

Because the agent is uncertainty averse, they will also take the worst-case scenario over $\theta \in K_\delta$. Thus, the division manager's problem becomes

$$U_{d,t} = \max_a \min_{\tilde{P}\in\mathcal{P}_{HQ}^{\Theta}\left(\left\{a_d, a_{d'}\right\}\right)} E_t^{a^*,\theta} u\left(w_d\left(h_1\right) - \int_0^1 c_d\left(\tilde{a}_{d,t}\right)dt\right). \tag{A21}$$

$$U_{d,t} = \max_a \min_{\theta_t\in K_\delta\left(a_t\right)} E_t^{a^*}\left[u\left(w_d\left(h_1\right) - \int_0^1 c_d\left(\tilde{a}_{d,t}\right)dt\right)\frac{dP^a}{dP^{a^*}}\frac{d\tilde{P}^{a,\theta}}{dP^a}\right]. \tag{A22}$$

Applying Girsanov's Theorem and rearranging, we can express

$$U_{d,t} = \max_a \min_{\theta_t\in K_\delta\left(a_t\right)} E_t^{a,\theta}\left[u\left(\tilde{W}\right)\psi_t\right], \tag{A23}$$

where, defining $H_d\left(a,\theta,B\right) \equiv B_{d,t}^{\prime}\mu^\theta\left(a_t\right) - c_d\left(a_{d,t}\right)$,

$$\tilde{W} = w_{d,0} + \int\left[H_d\left(a,\theta,B\right) - H_d\left(a^*,\theta,B\right)\right]dt, \tag{A24}$$

and $\psi_t = \exp\left(\int_0^t\left(-rB_{d,t}^{\prime}\right)dW_t^\theta - \int_0^t\left(-rB_{d,t}\right)^{\prime}\Sigma\left(-rB_{d,t}\right)dt\right)$ is a martingale. Therefore, the choice of a and θ affect $U_{d,t}$ only through $\tilde{W}$, and only through H. Therefore, the DM solves

$$\max_{a_t} \min_{\theta_t\in K_d\left(a_t\right)} H_d\left(a,\theta;B\right) \tag{A25}$$

for all points in time. Because a_d^* solves the maximization problem, $U_{d,t}$ is a supermartingale for any choice of a_d and a martingale only if $a_d = a_d^*$ almost surely. Therefore, a_d^* is the optimal effort choice. Note this holds for both division managers at every time.

Similarly, HQ solves

$$\Pi_{d,t} = \max_{\{w,a\}} \min_{\tilde{P}\in\mathcal{P}_{HQ}^{\Theta}\left(\left\{a_d,a_{d'}\right\}\right)} E\pi\left(X\right) \tag{A26}$$

where $X = Y_A + Y_B - w_A - w_B$. Let $\phi_t = 1 - B_{A,t} - B_{B,t}$. Similar to above, we can express

$$\Pi_{d,t} = \max_{\{w,a\}} \min_{\theta_t\in K_{HQ}\left(a_t\right)} E_t^{a,\theta}\left[\pi\left(\tilde{X}\right)\psi_t^{HQ}\right] \tag{A27}$$

where $\tilde{X} = \int H_{HQ}dt - w_{A,0} - w_{B,0}$,

$$\begin{aligned} H_{HQ}\left(a^*, B, \theta_{HQ}^*, \theta_d^*\right) \quad &= \quad \phi^{\prime}\mu^{\theta^*_{HQ}}\left(a_t^*\right) + H_A\left(a_A^*, \theta_A^*, B\right) + H_B\left(a_B^*, \theta_B^*, B\right) \\ &\quad -\frac{r}{2}B_{A,t}^{\prime}\Sigma B_{A,t} - \frac{r}{2}B_{B,t}^{\prime}\Sigma B_{B,t} - \frac{R}{2}\phi_t^{\prime}\Sigma\phi \end{aligned} \tag{A28}$$

and $\psi_t^{HQ} = \exp\left(\int_0^t\left(-R\phi_t^{\prime}\right)dW_t^\theta - \int_0^t\left(-R\phi_t\right)^{\prime}\Sigma\left(-R\phi_t\right)dt\right)$ is a martingale. Note that for all points in time and history, H_{HQ} depends only on the instantaneous contract, B, instantaneous effort, $a_{d,t}^*$, and belief distortions at time

t, $\theta_{\delta,t}$. Therefore, at each point in time, HQ solves

$$\max_{\{w,a\}} \min_{\theta \in K_{HQ}(a_t)} H_{HQ}\left(a, B, \theta_{HQ}, \theta_d\right) \tag{A29}$$

subject to a^* and θ_d^* solving (A25). Because this problem is identical throughout the tree, the solution must remain the same. Therefore, HQ optimally grants the same exposure B to DMs at all points in the tree, so the contract is linear. ■

Theorem A1 implies that we can restrict our analysis to affine incentive contracts with constant coefficients, as assumed in the body of the paper. Intuitively, the process $w(h_t)$ is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the bivariate Brownian motion W_t. Thus, similar to Holmström and Milgrom (1987), the martingale representation theorem ensures that it can be represented as an Itô process, which guarantees instantaneous linearity of incentive contracts. Translation invariance of CARA utility (precluding wealth effects) and IID uncertainty ensure that HQ and division managers face the same instantaneous optimization problem at every point in the tree, ensuring overall linearity. In the optimal contract, HQ grant division managers a constant share β_d of their own division, a constant exposure γ_d to the other division, inducing constant effort, $a_d = a_{d,t}$, and constant belief distortions, (θ_d, θ_{HQ}), for all t, for all $t \in [0,1]$.

Theorem A1 implies that the solution to the dynamic model is equivalent to the solution of a corresponding static problem where HQ offer only affine contracts that depend on division cash-flows. The static problem that corresponds to the dynamic model can then be written in certainty equivalent form, as follows. Letting $b_A \equiv (\beta_A, \gamma_A)'$, $b_B \equiv (\gamma_B, \beta_B)'$, $\phi \equiv (\phi_A, \phi_B) = (\mathbf{1} - b_A - b_B)'$, where $1 = (1,1)'$, HQ choose a pair of incentive contracts and action profiles, $\{w_d, a_d\}_{d\in\{A,B\}}$, that maximize their (instantaneous) certainty equivalent objective function, solving

$$\max_{\{w,a\}} \quad \min_{\theta^{HQ} \in K_{HQ}(a)} \pi^\theta \equiv \ \phi' \mu^{\theta_{HQ}}(a) - \frac{R}{2}\phi' \Sigma \phi - s_A - s_B, \tag{A30}$$

subject to the constraint that division managers maximize the certainty equivalent of their objective function

$$\max_{\tilde{a}_d} \min_{\theta \in K_d(a)} u_d^\theta \equiv s_d + b_d' \mu^{\theta_d}(\tilde{a}_d, a_{d'}) - \frac{r}{2} b_d' \Sigma b_d - c_d(a_d), \tag{A31}$$

and to the participation constraints

$$\min_{\theta_d \in K_d(a)} s_d + b_d' \mu^{\theta_d}(a) - \frac{r}{2} b_d' \Sigma b_d - c_d(a_d) \geq 0 \tag{A32}$$

for $d \in \{A, B\}$. Note that, absent uncertainty, $K_{HQ}(a) = K_d(a) = \{0\}$ and problem (A30) - (A32) collapses to the corresponding static problem of Holmström and Milgrom (1987). Further, because Theorem A1 shows that the optimal contract implements constant effort, a_t, and constant distortions θ_t, it is sufficient to consider $\theta \in K_d(a)$.

Finally, we must show that it is consistent with IID Ambiguity of Chen and Epstein (2002) to have uncertainty on firm productivity, q. Note that IID Ambiguity considers uncertainty on θ, not on q, so we must show that ambiguity on productivity q, is equivalent to considering IID ambiguity on θ. Define $\epsilon = (\epsilon_A, \epsilon_B) \equiv \Gamma\theta$, and let

$$K_\delta(a) = \left\{\theta \middle| \left[-\ln\left(1 - \frac{\sigma|\epsilon_A|}{a_A q_A}\right) - \ln\left(1 - \frac{\sigma|\epsilon_B|}{a_B q_B}\right)\right] \leq \eta_\delta\right\}, \tag{A33}$$

Note these expressions are not history dependent and provide a special case of IID uncertainty. The parameter η_δ reflects the degree of confidence in the reference probability P^a held by agent δ, for $\delta \in \{HQ, A, B\}$, where $\eta_\delta = 0$ indicates full confidence, and increasing uncertainty is characterized by greater η_δ.

In the optimization problem (A30) - (A32), division managers beliefs are determined by minimizing their objective function u_d^θ where, from (A10), the density generator process θ characterizes belief distortions affecting the drift, $\mu^\theta(a) = Qa - \Gamma\theta$, of the cash-flow process Y. For expositional simplicity, we will consider an affine transformation of the distorted beliefs θ, and define

$$\hat{Q}^\delta a \equiv Qa - \Gamma\theta \ \text{ with } \ \hat{Q}^\delta \equiv \begin{bmatrix} \hat{q}_A^\delta & 0 \\ 0 & \hat{q}_B^\delta \end{bmatrix}, \tag{A34}$$

where $\hat{q}_{d'}^\delta$ represents the (distorted) belief of $\delta \in \{A, B, HQ\}$ on the productivity of division $d' = \{A, B\}$. From (A34), the core belief set (A33), expressed in terms of belief distortions θ, induces a corresponding core belief set

$$K_{\delta}^{\hat{q}} \equiv \left\{ \hat{q}^{\delta} | \ln\left(\frac{1}{1-\left|\frac{\hat{q}_A^{\delta}-q_A}{q_A}\right|}\right) + \ln\left(\frac{1}{1-\left|\frac{\hat{q}_B^{\delta}-q_B}{q_B}\right|}\right) \leq \eta_{\delta} \right\}. \tag{A35}$$

expressed in terms of belief distortions on division productivity, $\hat{q}^{\delta}$. Note that the set $K_{\delta}^{\hat{q}}$ does not depend on a because $K_{\delta}(a)$ scales in effort a and, thus, from (A34) uncertainty directly affects effort productivity, q. The transformation (A34) leads to corresponding objective functions for division managers and HQ

$$\hat{u}_d \equiv s_d + b_d' \hat{Q}^d a - \frac{r}{2} b_d' \Sigma b_d - c_d(a_d), \tag{A36}$$

$$\hat{\pi} \equiv (\mathbf{1} - b_A - b_B)' \hat{Q}^{HQ} a - s_A - s_B. \tag{A37}$$

The following Lemma establishes the equivalency of considering core beliefs sets K_{δ} and $K_{\delta}^{\hat{q}}$.

Lemma A1 *The following two problems are equivalent:*

$$\min_{\theta \in K_d} u_d^{\theta} = \min_{\hat{q}^d \in K_d^{\hat{q}}} \hat{u}_d, \tag{A38}$$

where $\hat{q}^d \equiv (\hat{q}_A^d, \hat{q}_B^d)$*, with* $\delta \in \{A, B\}$*. Similarly, for HQ* $\min_{\theta \in K_{HQ}(a)} \pi^{\theta}$ *is equivalent to* $\min_{\hat{q}^{HQ} \in K_{HQ}^{\hat{q}}} \hat{\pi}$.

Proof of Lemma A1. For $\theta_{\delta} \in K_{\delta}(a)$, define $\hat{q}^{\delta}$ so that $\hat{q}_d^{\delta} \equiv q_d - \frac{1}{a_d}(\Gamma\theta)_d$, an affine transformation of θ_{δ}, so that $\mu^{\theta_{\delta}} = Qa - \Gamma\theta_{\delta} = \hat{Q}^{\delta} a$. (A35) follows by substituting $\hat{q}^{\delta}$ into (A33): $\theta_{\delta} \in K_{\delta}(a)$ iff $\hat{q}^{\delta} \in K_{\delta}^{\hat{q}}$. ■

B Internet Appendix B: Proofs

Proof of Theorem 1. Linearity follows from Theorem A1, by setting $K_A = K_B = K_{HQ} = \{0\}$; thus compensation contract to division manager d is $w_d = s_d + \beta_d Y_{d,1} + \gamma_d Y_{d',1}$. Substituting for μ^{θ} and Σ in (A13), division manager d selects a_d to solve

$$\max_{a_d} \ u_d = s_d + \beta_d q_d a_d + \gamma_d q_{d'} a_{d'} - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - c_d(a_d). \tag{B1}$$

Because u_d is strictly concave, the incentive constraint is fully characterized by the first-order condition and the unique maximizer is $a_d = \beta_d Z_d q_d$. Because of translation invariance of u_d, (A14) always binds at an optimum, giving

$$s_d = \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) + c_d(a_d) - \beta_d q_d a_d - \gamma_d q_{d'} a_{d'}. \tag{B2}$$

Substituting for s_d into HQ objective, (A12), we obtain

$$\hat{\pi} = \sum_{d \in \{A,B\}} \left[q_d a_d - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - c_d(a_d) \right], \tag{B3}$$

Substituting for $a_d = \beta_d Z_d q_d$ in $\hat{\pi}$ and differentiating we obtain that

$$\beta_d = \frac{1}{1 + r\sigma^2(1-\rho^2)/(Z_d q_d^2)}, \quad \text{and} \quad \gamma_d = -\rho\beta_d. \tag{B4}$$

Second order conditions are satisfied by concavity of (A12). ■

Proof of Lemma 1. Division managers determine $\left(\hat{q}_d^d, \hat{q}_{d'}^d\right)$ in (14). We will focus on two cases: we start with the case where $\gamma_d \geq 0$, and then we consider the case $\gamma_d < 0$. Consider $\tilde{q}_d^d = q_d + \delta$, for $\delta > 0$. Switching to $\tilde{q}_d^{d-} = q_d - \delta$ lowers $\hat{u}_d$ by $2\beta_d a_d \delta$ while leaving the constraint unchanged. Therefore, it must be that $\hat{q}_d^d \leq q_d$. Similarly, switching from $\tilde{q}_{d'}^d = q_{d'} + \delta$, for $\delta > 0$ to $\tilde{q}_{d'}^{d-} = q_{d'} - \delta$ lowers $\hat{u}_d$ by $2\gamma_d a_{d'} \delta$, leaving the constraint unchanged. Therefore, it must also be that $\hat{q}_{d'}^d \leq q_{d'}$. Thus, we can express the Lagrangian as

$$\mathcal{L} \equiv -\hat{u}_d - \lambda\left[g_c - \eta_d\right] - \tau_d\left(\hat{q}_d^d - q_d\right) - \tau_{d'}\left(\hat{q}_{d'}^d - q_{d'}\right) \tag{B5}$$

where $g_c \equiv \ln \frac{q_d}{\hat{q}_d^d} + \ln \frac{q_{d'}}{\hat{q}_{d'}^d}$. Because problem (14) admits corner solutions, we characterize its solution by use of the full Kuhn-Tucker conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} - \lambda \frac{\partial g_c}{\partial \hat{q}_d^d} - \tau_d = -\beta_d a_d + \frac{\lambda}{\hat{q}_d^d} - \tau_d = 0, \\
\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} &= -\frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} - \lambda \frac{\partial g_c}{\partial \hat{q}_{d'}^d} - \tau_{d'} = -\gamma_d a_{d'} + \frac{\lambda}{\hat{q}_{d'}^d} - \tau_{d'} = 0, \\
& \lambda (g_c - \eta_d) + \tau_d \left(\hat{q}_d^d - q_d \right) + \tau_{d'} \left(\hat{q}_{d'}^d - q_{d'} \right) = 0, \\
\lambda &\geq 0,\ \tau_{d'} \geq 0,\ \tau_d \geq 0,\ \eta_d - g_c \geq 0,\ q_d - \hat{q}_d^d \geq 0,\ q_{d'} - \hat{q}_{d'}^d \geq 0.
\end{aligned} \tag{B6}$$

Note first that, from the definition of g_c, to satisfy the constraint $\eta_d - g_c \geq 0$ it must be $\hat{q}_d^d > 0$ and $\hat{q}_{d'} > 0$, which implies that $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = \frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$. Note also that $\beta_d a_d > 0$ implies that $\lambda > 0$, and thus that $g_c - \eta_d = 0$. In addition, it cannot be that both $\tau_d > 0$ and $\tau_{d'} > 0$ because, if so, then $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = q_{d'}$, which would imply that $g_c = 0 < \eta_d$, which contradicts $\lambda > 0$. This leaves us with three types of solutions: $\tau_d = \tau_{d'} = 0$, $\tau_d > 0 = \tau_{d'}$, and $\tau_d = 0 < \tau_{d'}$.

If $\tau_d = \tau_{d'} = 0$, then $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = \frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$ together imply that $\lambda = \beta_d a_d \hat{q}_d^d$ and $\lambda = \gamma_d a_{d'} \hat{q}_{d'}^d$, giving $\beta_d a_d \hat{q}_d^d = \gamma_d a_{d'} \hat{q}_{d'}^d$. Because $g_c = \eta_d$ implies that $\hat{q}_d^d \hat{q}_{d'}^d = e^{-\eta_d} q_d q_{d'}$, after substitution this implies that $\frac{\beta_d a_d}{\gamma_d a_{d'}} \left(\hat{q}_d^d \right)^2 = e^{-\eta_d} q_d q_{d'}$, or equivalently, $\hat{q}_d^d = \left[e^{-\eta_d} H_d \right]^{\frac{1}{2}} q_d$, where $H_d = \frac{\gamma_d a_{d'} q_{d'}}{\beta_d a_d q_d}$. Similarly, $\hat{q}_{d'}^d = \left[e^{-\eta_d} \frac{1}{H_d} \right]^{\frac{1}{2}} q_{d'}$. In order for this to be feasible, however, it must be that $\hat{q}_d^d \leq q_d$, or equivalently, $H_d \leq e^{\eta_d}$, and $\hat{q}_{d'}^d \leq q_{d'}$, or equivalently, $H_d \geq e^{-\eta_d}$, giving case (ii). If $\tau_d > 0 = \tau_{d'}$, then $\hat{q}_d^d = q_d$ and, from $g_c = \eta_d$, also $\hat{q}_{d'}^d = e^{-\eta_d} q_{d'}$. Note that $\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$ implies that $\lambda = \gamma_d a_{d'} e^{-\eta_d} q_{d'}$ and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = 0$, we have that

$$\tau_d = -\beta_d a_d + \frac{\gamma_d a_{d'} e^{-\eta_d} q_{d'}}{q_d} = \beta_d a_d \left(H_d e^{-\eta_d} - 1 \right) > 0, \tag{B7}$$

which requires $H_d > e^{\eta_d}$, giving case (i). Finally, if $\tau_d = 0 < \tau_{d'}$, then $\hat{q}_{d'}^d = q_{d'}$ and, from $g_c = \eta_d$, also $\hat{q}_d^d = e^{-\eta_d} q_d$. Note that now $\frac{\partial \mathcal{L}}{\partial \hat{q}_d^d} = 0$ implies that $\lambda = \beta_d a_d e^{-\eta_d} q_d$, and, from $\frac{\partial \mathcal{L}}{\partial \hat{q}_{d'}^d} = 0$, we have that

$$\tau_{d'} = -\gamma_d a_{d'} + \frac{\beta_d a_d e^{-\eta_d} q_d}{q_{d'}} = \gamma_d a_{d'} \left(H_d^{-1} e^{-\eta_d} - 1 \right) \geq 0, \tag{B8}$$

which requires $0 \leq H_d < e^{-\eta_d}$, giving part of case (iii).

The case with $\gamma_d < 0$ proceeds in a similar way, giving cases (iv), (v) and the remainder of case (iii), and is omitted. Note that in the case of interior beliefs, case (iv), for $H_d \in (-e^{\eta_d}, -e^{-\eta_d})$ we have

$$\hat{q}_d^d = \left[e^{-\eta_d} |H_d| \right]^{\frac{1}{2}} q_d, \text{ and } \hat{q}_{d'}^d = \left(2 - \left[e^{-\eta_d} |H_d|^{-1} \right]^{\frac{1}{2}} \right) q_{d'}. \tag{B9}$$

Finally, in case (v) we have $\hat{q}_{d'}^d = q_{d'}$ and $\hat{q}_{d'}^d = (2 - e^{-\eta_d}) q_{d'}$ for $H_d \leq -e^{\eta_d}$. ■

Proof of Lemma 2. The lemma is shown in two steps. First, we obtain division managers' best response functions, $a_d = Z_d \beta_d \hat{q}_d^d$, as function of their beliefs, as in Lemma 1. Second, because $\hat{q}_d^d$ is positive, continuous, and increasing in $a_{d'}$, we characterize the Nash equilibrium in terms of $\log(a_d)$ and we apply the contraction mapping theorem, proving uniqueness.

Division manager $d \in \{A, B\}$ chooses effort level a_d to solve (16) by setting

$$\frac{d}{da_d} \hat{u}_d(a, \hat{q}_d^d(a, w)) = \frac{\partial \hat{u}_d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} \frac{\partial \hat{q}_d^d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} \frac{\partial \hat{q}_{d'}^d}{\partial a_d} = \frac{\partial \hat{u}_d}{\partial a_d} = 0, \tag{B10}$$

where the second equality holds by the envelope theorem, as follows. For cases (ii) and (iv) of Lemma 1, we have that $\frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} = \lambda \frac{\partial g}{\partial \hat{q}_d}$ and $\frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} = \lambda \frac{\partial g}{\partial \hat{q}_{d'}}$, giving

$$\frac{\partial \hat{u}_d}{\partial \hat{q}_d^d} \frac{\partial \hat{q}_d^d}{\partial a_d} + \frac{\partial \hat{u}_d}{\partial \hat{q}_{d'}^d} \frac{\partial \hat{q}_{d'}^d}{\partial a_d} = \lambda \left(\frac{\partial g}{\partial \hat{q}_d} \frac{\partial \hat{q}_d^d}{\partial a_d} + \frac{\partial g}{\partial \hat{q}_{d'}} \frac{\partial \hat{q}_{d'}^d}{\partial a_d} \right) = \lambda \frac{dg}{da_d} = 0 \tag{B11}$$

because $g = e^{-\eta_d}$. In cases (i)-(iii)-(v), $\hat{q}_d^d$ and $\hat{q}_{d'}^d$ do not depend on a_d, and $\frac{\partial \hat{q}_d^d}{\partial a_d} = \frac{\partial \hat{q}_{d'}^d}{\partial a_d} = 0$, giving $\frac{d \hat{u}_d}{d a_d} = \frac{\partial \hat{u}_d}{\partial a_d} = \beta_d \hat{q}_d^d - \frac{a_d}{Z_d} = 0$.

Thus, the best response functions are $a_d = Z_d \beta_d \hat{q}_d^d$, where beliefs $\hat{q}_d^d$ are from Lemma 1. If $\gamma_d = 0$, we have that $H_d = 0$, giving $a_d = Z_d \beta_d e^{-\eta_d} q_d$. If $\gamma_d \neq 0$, the best response depends on the effort by the other division manager, $a_{d'}$. If the other division manager, $d' \neq d$, exerts low effort $a_{d'} < a_{d'}^L \equiv \frac{Z_d \beta_d^2 e^{-2\eta_d} q_d^2}{|\gamma_d| q_{d'}}$, we have that $|H_d| < e^{-\eta_d}$ and division manager d holds pessimistic belief as in case (iii) of Lemma 1, $\hat{q}_d^d = e^{-\eta_d} q_d$, giving $a_d = a_d^{1*} \equiv Z_d \beta_d e^{-\eta_d} q_d$. If division manager d' exerts moderate level of effort, $a_{d'}^L \leq a_{d'} < a_{d'}^H \equiv \frac{Z_d \beta_d^2 e^{\eta_d} q_d^2}{|\gamma_d| q_{d'}}$, division manager d hold beliefs as in case (ii) of Lemma 1, if $\gamma_d > 0$, and as in case (iv), if $\gamma_d < 0$; thus $|H_d| \in \left[e^{-\eta_d}, e^{\eta_d}\right]$ and $a_d = \left[Z_d^2 |\gamma_d| a_{d'} \beta_d e^{-\eta_d} q_{d'} q_d\right]^{\frac{1}{3}}$. Finally, if division manager d' exerts a high level of effort, $a_{d'} > a_{d'}^H$, division manager d hold beliefs as in case (i) of Lemma 1, if $\gamma_d > 0$, and as in case (v), if $\gamma_d < 0$; thus $|H_d| > e^{\eta_d}$ and $a_d = Z_d \beta_d q_d$. The best response function for DM d is therefore given by

$$a_d^*(a_{d'}) = \begin{cases} a_d^{1*} \equiv Z_d \beta_d e^{-\eta_d} q_d & a_{d'} < a_{d'}^L \\ \tilde{a}_d^*(a_{d'}) \equiv \left[Z_d^2 |\gamma_d| a_{d'} \beta_d e^{-\eta_d} q_{d'} q_d\right]^{\frac{1}{3}} & a_{d'}^L \leq a_{d'} \leq a_{d'}^H \\ a_d^{2*} \equiv Z_d \beta_d q_d & a_{d'} > a_{d'}^H \end{cases}. \tag{B12}$$

A Nash equilibrium is a pair $\{a_A, a_B\}$ such that $a_d = a_d^*(a_{d'})$, $d \in \{A, B\}$, $d \neq d'$. Note that $a_d^*(a_{d'})$ is a positive, continuous, and increasing function of $a_{d'}$. Expressing the best response in logs, we obtain

$$\ln a_d^*(\ln a_{d'}) = \begin{cases} \ln Z_d \beta_d e^{-\eta_d} q_d & \ln a_{d'} < \ln a_{d'}^L \\ \ln\left[Z_d^2 |\gamma_d| \beta_d e^{-\eta_d} q_{d'} q_d\right]^{\frac{1}{3}} + \frac{1}{3}\ln(a_{d'}) & \ln a_{d'}^L \leq \ln a_{d'} \leq \ln a_{d'}^H \\ \ln Z_d \beta_d q_d & \ln a_{d'} > \ln a_{d'}^H \end{cases}. \tag{B13}$$

Further, note $\frac{d \ln a_d^*}{d \ln a_{d'}} = 0$ for $a_{d'} < a_{d'}^L$ and $a_{d'} > a_{d'}^H$, while $\frac{d \ln a_d^*}{d \ln a_{d'}} = \frac{1}{3}$ for $a_{d'}^L < a_{d'} < a_{d'}^H$. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ so that $F \equiv (\ln a_A^*(\ln a_B), \ln a_B^*(\ln a_A))'$, and let $d(x, y)$ be the Euclidean distance. For $x, y \in \mathbb{R}^2$, define $\tilde{x}_d \equiv \max\left\{\ln a_d^L, \min\left\{x_d, \ln a_d^H\right\}\right\}$ and $\tilde{y}_d \equiv \max\left\{\ln a_d^L, \min\left\{y_d, \ln a_d^H\right\}\right\}$, we have

$$\begin{aligned} d(F(x), F(y)) &= \sqrt{(\ln a_A^*(x_B) - \ln a_A^*(y_B))^2 + (\ln a_B^*(x_A) - \ln a_B^*(y_A))^2} \\ &= \sqrt{(\ln a_A^*(\tilde{x}_B) - \ln a_A^*(\tilde{y}_B))^2 + (\ln a_B^*(\tilde{x}_A) - \ln a_B^*(\tilde{y}_A))^2} \\ &= \sqrt{\left[\frac{1}{3}(\tilde{x}_B - \tilde{y}_B)\right]^2 + \left[\frac{1}{3}(\tilde{x}_A - \tilde{y}_A)\right]^2} = \frac{1}{3} d(\tilde{x}, \tilde{y}) \leq \frac{1}{3} d(x, y), \end{aligned} \tag{B14}$$

which implies that $0 \leq d(F(x), F(y)) \leq \frac{1}{3} d(x, y)$ for all $x, y \in \mathbb{R}^2$. Thus, F is a contraction mapping and the Nash Equilibrium exists and is unique.

Because the best-response function is constant if d' exerts low effort, $a_{d'} < a_{d'}^L$, and if d' exerts high effort, $a_{d'} > a_{d'}^H$, the Nash Equilibrium is fully determined. All that remains to be determined is the Nash Equilibrium effort for d when $a_{d'}^L \leq a_{d'} \leq a_{d'}^H$. There are three possible cases:

(1) If $a_{d'} = a_{d'}^{1*} > a_{d'}^L$, so that $|H_{d'}| \leq e^{-\eta_{d'}}$, then

$$a_d = \tilde{a}_d^*\left(a_{d'}^{1*}\right) = \left[Z_d^2 Z_{d'} e^{-(\eta_d + \eta_{d'})} |\gamma_d| \beta_{d'} \beta_d q_{d'}^2 q_d\right]^{\frac{1}{3}}; \tag{B15}$$

(2) If $a_{d'} = a_{d'}^{2*} < a_{d'}^H$, so that $|H_{d'}| \geq e^{\eta_{d'}}$, then

$$a_d = \tilde{a}_d^*\left(a_{d'}^{2*}\right) = \left[Z_d^2 Z_{d'} e^{-\eta_d} |\gamma_d| \beta_{d'} \beta_d q_{d'}^2 q_d\right]^{\frac{1}{3}}; \tag{B16}$$

(3) if $a_{d'}^{1*} < a_{d'} < a_{d'}^{2*}$, so that $|H_{d'}| \in \left(e^{-\eta_{d'}}, e^{\eta_{d'}}\right)$, then setting $a_d = \tilde{a}_d^*(a_{d'})$ and $a_{d'} = \tilde{a}_{d'}^*(a_d)$, after solving we obtain

$$a_d = \check{a}_d \equiv \left[e^{-\eta_d} Z_d^2 \beta_d |\gamma_d|\right]^{\frac{3}{8}} \left[e^{-\eta_{d'}} Z_{d'}^2 \beta_{d'} |\gamma_{d'}|\right]^{\frac{1}{8}} \left[q_d q_{d'}\right]^{\frac{1}{2}}. \tag{B17}$$

Comparative statics follow by differentiation. ■

Proof of Theorem 2. Because (19) binds and $r = 0$, HQ payoff $\hat{\pi}$ is now equal to

$$\hat{\pi} = \sum_{\substack{d, d' \in \{A, B\}, \\ d' \neq d}} \left(q_d a_d - \beta_d a_d \left(q_d - \hat{q}_d^d\right) - \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) - \frac{a_d^2}{2 Z_d}\right), \tag{B18}$$

where a_d are the Nash-equilibrium effort levels of Lemma 2. The proof is in two steps. First, we show that $\hat{\pi}$ is symmetric in γ_d around zero; in the second step, we find the optimal contract under the restriction that $\gamma_d \geq 0$.

Note that, from Lemma 1, $\hat{q}_d^d$ depends on γ_d only through its absolute value, $|\gamma_d|$. Thus, from Lemma 2, equilibrium action $a_d = \beta_d Z_d \hat{q}_d^d$ also depends on $|\gamma_d|$ only. This implies the first term of the disagreement discount, $\beta_d a_d \left(q_d - \hat{q}_d^d\right)$, depends only on $|\gamma_d|$. We next show that, if $\gamma_d < 0$, the second term of the disagreement discount, $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right)$, is unchanged by offering cross pay, $|\gamma_d|$, rather than relative performance evaluation, $\gamma_d < 0$. From Lemma 1, let $\hat{q}_{d'}^{d+}$ be the belief held by the DM when receiving $|\gamma_d|$ instead of $\gamma_d < 0$. We will show $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = |\gamma_d| \, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right)$. Consider in turn cases (iii), (iv) and (v) in Lemma 1.

First, in case (v) we have that $H_d < -e^{\eta_d}$ and $\hat{q}_{d'}^d = \left(2 - e^{-\eta_d}\right) q_{d'}$. This implies that replacing γ_d with $|\gamma_d|$ gives that $|H_d| > e^{\eta_d}$ and beliefs will be as in case (i). Thus, setting $\hat{q}_{d'}^{d+} = e^{-\eta_d} q_{d'}$ we obtain

$$|\gamma_d| \, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right) = |\gamma_d| \, a_{d'} \left(1 - e^{-\eta_d}\right) q_{d'} = \gamma_d a_{d'} \left(e^{-\eta_d} - 1\right) q_{d'} = \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right). \tag{B19}$$

In case (iii), we have that $|H_d| < e^{-\eta_d}$. This implies that $\hat{q}_{d'}^{d+} = \hat{q}_{d'}^d = q_{d'}$, so

$$|\gamma_d| \, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right) = \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = 0. \tag{B20}$$

In case (iv), $H_d \in \left(-e^{\eta_d}, -e^{-\eta_d}\right)$ and $\hat{q}_{d'}^d = \left(2 - \left[e^{-\eta_d} \frac{\beta_d a_d q_d}{|\gamma_d| a_{d'} q_{d'}}\right]^{\frac{1}{2}}\right) q_{d'}$, giving

$$\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = \gamma_d a_d \left(\left[\frac{e^{-\eta_d} \beta_d a_d q_d}{|\gamma_d| \, a_{d'} q_{d'}}\right]^{\frac{1}{2}} - 1\right) q_{d'} = |\gamma_d| \, a_{d'} \left(1 - \left[\frac{e^{-\eta_d} \beta_d a_d q_d}{|\gamma_d| \, a_{d'} q_{d'}}\right]^{\frac{1}{2}}\right) q_{d'}. \tag{B21}$$

This implies that replacing γ_d with $|\gamma_d|$, beliefs will be as in case (ii). Thus, setting $\hat{q}_{d'}^{d+} = \left[e^{-\eta_d} \frac{\beta_d a_d q_d}{|\gamma_d| a_{d'} q_{d'}}\right]^{\frac{1}{2}} q_{d'}$ we obtain

$$|\gamma_d| \, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right) = \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right). \tag{B22}$$

Therefore, $\hat{\pi}(\gamma_d) = \hat{\pi}(|\gamma_d|)$ and $\hat{\pi}$ is symmetric in γ_d around zero.

Because HQ is indifferent between $|\gamma_d|$ and γ_d, it is sufficient to consider $\gamma_d \geq 0$. If $\gamma_d > e^{\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division manager beliefs are in case (i) of Lemma 1, with $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = e^{-\eta_d} q_{d'}$, giving $a_d = \beta_d Z_d q_d$. Thus, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'} q_{d'} \left(1 - e^{-\eta_d}\right) < 0$, and setting $\gamma_d > e^{\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ is not optimal. Similarly, if $\gamma_d < e^{-\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division manager beliefs are in case (iii) of Lemma 1, with $\hat{q}_d^d = e^{-\eta_d} q_d$ and $\hat{q}_{d'}^d = q_{d'}$, giving $a_d = \beta_d Z_d e^{-\eta_d} q_d$. In addition, $\hat{q}_{d'}^d = q_{d'}$ and $\hat{q}_d^d = e^{-\eta_d} q_d$ together imply that $\frac{\partial \hat{\pi}}{\partial \gamma_d} = 0$ and it is weakly optimal to set $\gamma_d \geq e^{-\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$. This implies that HQ set $e^{-\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}} \leq \gamma_d \leq e^{\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ and induce beliefs that are in case (ii) of 1, with $H_d \in \left[e^{-\eta_d}, e^{\eta_d}\right]$.

Because the participation constraint binds, HQ objective function becomes

$$\hat{\pi} = (\mathbf{1} - b_A - b_B)' \, Q \check{a}_d + \left(\hat{u}_A(\check{a}_A, \hat{q}^A) - s_A\right) + \left(\hat{u}_B(\check{a}_B, \hat{q}^B) - s_B\right). \tag{B23}$$

where $\hat{u}_d(\check{a}_d, \hat{q}^d) = \min_{\hat{q}^d \in K_d^{\hat{q}}} \hat{u}_d$, with $\hat{u}_d = s_d + \beta_d \check{a}_d \hat{q}_d^d + \gamma_d \check{a}_{d'} \hat{q}_{d'}^d - \frac{\check{a}_d^2}{2Z_d} = 0$ and where $\check{a}_d$ is the Nash equilibrium given by (B17) in the proof of Lemma 2. This implies that

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -q_d \check{a}_d + \left(1 - \beta_d - \gamma_{d'}\right) q_d \frac{\partial \check{a}_d}{\partial \beta_d} + \left(1 - \beta_{d'} - \gamma_d\right) q_{d'} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned} \tag{B24}$$

Because $\frac{\partial \hat{u}_d}{\partial \beta_d} = \check{a}_d \hat{q}_d^d$, $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}_{d'}^d$, and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, by applying the envelope theorem on $\hat{u}_d(\check{a}_d, \hat{q}^d)$, we obtain that

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} = \frac{\partial \hat{u}_d}{\partial \beta_d} + \frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} \frac{\partial \check{a}_{d'}}{\partial \beta_d} = \check{a}_d \hat{q}_d^d + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d}. \tag{B25}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \beta_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'} \hat{q}_d^{d'}$, and $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d} = \frac{\partial \hat{u}_{d'}}{\partial \beta_d} + \frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} \frac{\partial \check{a}_d}{\partial \beta_d} = \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\beta_d}. \tag{B26}$$

Together, (B25) and (B26) give that

$$\frac{d\hat{\pi}}{d\beta_d} = -\check{a}_d\left(q_d-\hat{q}_d^d\right)+\left(1-\beta_d-\gamma_{d'}\right)q_d\frac{3\check{a}_d}{8\beta_d}+\left(1-\beta_{d'}-\gamma_d\right)q_{d'}\frac{\check{a}_{d'}}{8\beta_d} +\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\beta_d}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\beta_d}. \tag{B27}$$

Consider now γ_d. We have that

$$\frac{d\hat{\pi}}{d\gamma_d} = -q_{d'}\check{a}_{d'}+\left(1-\beta_d-\gamma_{d'}\right)q_d\frac{\partial\check{a}_d}{\partial\gamma_d}+\left(1-\beta_{d'}-\gamma_d\right)q_{d'}\frac{\partial\check{a}_{d'}}{\partial\gamma_d} +\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\gamma_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d}. \tag{B28}$$

Because $\frac{\partial\hat{u}_d}{\partial\gamma_d}=\check{a}_{d'}\hat{q}_{d'}^d$, $\frac{\partial\hat{u}_d}{\partial\check{a}_{d'}}=\gamma_d\hat{q}_{d'}^d$, and $\frac{\partial\check{a}_{d'}}{\partial\gamma_d}=\frac{\check{a}_{d'}}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_d(\check{a}_d,\hat{q}^d)$, we obtain that

$$\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(a_d,w_d))}{d\gamma_d}=\frac{\partial\hat{u}_d}{\partial\gamma_d}+\frac{\partial\hat{u}_d}{\partial\check{a}_{d'}}\frac{\partial\check{a}_{d'}}{\partial\gamma_d}=\check{a}_{d'}\hat{q}_{d'}^d+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\gamma_d}. \tag{B29}$$

Similarly, because $\frac{\partial\hat{u}_{d'}}{\partial\gamma_d}=0$, $\frac{\partial\hat{u}_{d'}}{\partial\check{a}_d}=\gamma_{d'}\hat{q}_d^{d'}$, and $\frac{\partial\check{a}_d}{\partial\gamma_d}=\frac{3\check{a}_d}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d}=\frac{\partial\hat{u}_{d'}}{\partial\gamma_d}+\frac{\partial\hat{u}_{d'}}{\partial\check{a}_d}\frac{\partial\check{a}_d}{\partial\gamma_d}=\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\gamma_d}. \tag{B30}$$

Together, (B29) and (B30) give that

$$\frac{d\hat{\pi}}{d\gamma_d} = -\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)+\left(1-\beta_d-\gamma_{d'}\right)q_d\frac{3\check{a}_d}{8\gamma_d}+\left(1-\beta_{d'}-\gamma_d\right)q_{d'}\frac{\check{a}_{d'}}{8\gamma_d} +\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\gamma_d}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\gamma_d}. \tag{B31}$$

Thus, from (B27) and (B31) we obtain the first-order conditions:

$$\frac{d\hat{\pi}}{d\beta_d}=-\check{a}_d\left(q_d-\hat{q}_d^d\right)+\frac{\Delta_d}{\beta_d}=0;\quad \frac{d\hat{\pi}}{d\gamma_d}=-\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)+\frac{\Delta_d}{\gamma_d}=0, \tag{B32}$$

where $\Delta_d\equiv\left(1-\beta_d-\gamma_{d'}\right)q_d\frac{3\check{a}_d}{8}+\left(1-\beta_{d'}-\gamma_d\right)q_{d'}\frac{\check{a}_{d'}}{8}+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(q_d-\hat{q}_d^d\right)=\gamma_{d'}\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right). \tag{B33}$$

Because, from Lemma 1, $\beta_d\check{a}_d\hat{q}_d^d=\gamma_d\check{a}_{d'}\hat{q}_{d'}^d$, we have that (B33) implies that $\beta_d\check{a}_dq_d=\gamma_d\check{a}_{d'}q_{d'}$ and thus that $H_d=1$, leading to $\hat{q}_d^d=\hat{q}_{d'}^d=e^{-\frac{\eta_d}{2}}q_d$ and $\check{a}_d=e^{-\frac{\eta_d}{2}}\beta_dZ_dq_d$. Substituting the values of γ_d and $\check{a}_d$ into HQ objective, we obtain

$$\hat{\pi}=\sum_{\substack{d,d'\in\{A,B\},\\ d'\neq d}}\left[\beta_dZ_dq_d\hat{q}_d^d-2\beta_d^2Z_d\hat{q}_d^d\left(q_d-\hat{q}_d^d\right)-\frac{\beta_d^2Z_d\left(\hat{q}_d^d\right)^2}{2}\right], \tag{B34}$$

Differentiating, we obtain

$$\frac{d\hat{\pi}}{d\beta_d}=Z_dq_d\hat{q}_d^d-4\beta_dZ_d\hat{q}_d^d\left(q_d-\hat{q}_d^d\right)-\beta_dZ_d\left(\hat{q}_d^d\right)^2=0, \tag{B35}$$

giving

$$\beta_d=\frac{1}{1+3\left(1-\hat{q}_d^d/q_d\right)}. \tag{B36}$$

Finally, setting $H_d=1$ gives

$$\gamma_d=\frac{\check{a}_dq_d}{\check{a}_{d'}q_{d'}}\beta_d=\xi_d\beta_d,\ \text{ where } \xi_d\equiv\frac{\check{a}_dq_d}{\check{a}_{d'}q_{d'}}. \tag{B37}$$

Substituting for the values of $\check{a}_d$ and $\check{a}_{d'}$, given the expression for beliefs in Lemma 1, we obtain

$$\xi_d=\frac{1-3\left(1-\hat{q}_{d'}^{d'}/q_{d'}\right)}{1-3\left(1-\hat{q}_d^d/q_d\right)}\frac{\hat{q}_d^d/q_d}{\hat{q}_{d'}^{d'}q_{d'}}\frac{Z_dq_d^2}{Z_{d'}q_{d'}^2}. \tag{B38}$$

If HQ implement the symmetric contract, with $\gamma_d=-\frac{\check{a}_dq_d^d}{\check{a}_{d'}q_{d'}^d}\beta_d$, we obtain that $\hat{q}_{d'}^d=\left(2-e^{-\frac{\eta_d}{2}}\right)q_{d'}$. Thus $|\gamma_d|=$

$\xi_d \beta_d$. If divisions are symmetric, and condition (S) holds, $\xi_d = 1$. Comparative statics follow by direct differentiation. ■

Proof of Theorem 3. Because the participation constraint (19) binds, HQ payoff, $\hat{\pi}$, now is equal to

$$\sum_{\substack{d,d' \in \{A,B\} \\ d' \neq d}} \left[(1 - \beta_d - \gamma_{d'}) q_d a_d + \beta_d a_d \hat{q}_d^d + \gamma_d a_{d'} \hat{q}_{d'}^d - \frac{a_d^2}{2Z_d} - \frac{r\sigma^2 \left(\beta_d^2 + 2\beta_d \gamma_d \rho + \gamma_d^2\right)}{2} \right] \tag{B39}$$

where $\{a_A, a_B\}$ are the Nash equilibrium effort levels of Lemma 2.

Different from the case of Theorem 2, because of the presence of the last term, HQ objective function $\hat{\pi}$ admits multiple strict local maxima. The proof therefore proceeds in two steps. First, we consider candidate optimal contracts that induce division managers to hold one of four possible configurations of beliefs (implied by Lemma 1). Specifically, we consider contracts as follows. Case (A): a small exposure to the other division leading to $|H_d| < e^{-\eta_d}$, corresponding to case (iii) of Lemma 1; Case (B): a moderate positive exposure to the other division, leading to $H_d \in \left(e^{-\eta_d}, e^{\eta_d}\right)$, corresponding to case (ii) of Lemma 1; Case (C): a moderate negative exposure to the other division, leading to $H_d \in \left(-e^{\eta_d}, -e^{-\eta_d}\right)$, corresponding to case (iv) of Lemma 1; Case (D): a large (negative or positive) exposure to the other division, leading to $|H_d| > e^{\eta_d}$ corresponding to cases (i) and (v) of Lemma 1. Second, we compare payoffs to HQ from optimal contracts in these regions and we determine the globally optimal contract.

Case (A): If $|H_d| < e^{-\eta_d}$, have $\hat{q}_d^d = e^{-\eta_d} q_d$ and $\hat{q}_{d'}^d = q_{d'}$, which do not depend on γ_d. Similarly, by Lemma 2, $a_d = \beta_d Z_d e^{-\eta_d} q_d$, which does not depend on γ_d as well. Therefore, setting

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = -r\sigma^2 \left(\rho \beta_d + \gamma_d\right) = 0 \tag{B40}$$

gives $\gamma_d = -\rho \beta_d$ and γ_d is set to hedge risk with no effect on incentives. Substituting in $\hat{\pi}$ and differentiating we obtain

$$\frac{\partial \hat{\pi}}{\partial \beta_d} = \left(1 - 2\beta_d\right) Z_d q \hat{q}_d^d + \beta_d Z_d \left(\hat{q}_d^d\right)^2 - r\sigma^2 \beta_d \left(1 - \rho^2\right) \tag{B41}$$

Therefore

$$\beta_d^1 \equiv \frac{1}{1 + \left(1 - \hat{q}_d^d / q\right) + r\sigma^2 \left(1 - \rho^2\right) / (Zq\hat{q}_d^d)}. \tag{B42}$$

After substitution, this gives HQ payoff under condition (S)

$$\hat{\pi}^1 \equiv \frac{\left[e^{-\eta} Z q^2\right]^2}{\left(2 - e^{-\eta}\right) e^{-\eta} Z q^2 + r\sigma^2 \left(1 - \rho^2\right)}. \tag{B43}$$

Case (B): If $H_d \in \left(e^{-\eta}, e^{\eta}\right)$, we can express the payoff to HQ as

$$\hat{\pi} = (\mathbf{1} - b_A - b_B)' Q a + \left(\hat{u}_A(a_A, \hat{q}^A(a_A, w_A)) - s_A\right) + \left(\hat{u}_B(a_B, \hat{q}^B(a_B, w_B)) - s_B\right), \tag{B44}$$

where $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \min_{\hat{q}^d \in K_d^{\hat{q}}} \hat{u}_d$, with

$$\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \beta_d a_d \hat{q}_d^d + \gamma_d a_{d'} \hat{q}_{d'}^d - \frac{r\sigma^2}{2} \left(\beta_d^2 + 2\rho \beta_d \gamma_d + \gamma_d^2\right) - \frac{a_d^2}{2Z_d} = 0, \tag{B45}$$

and where $\check{a}_d$ is the Nash equilibrium given by (B17). Because $\hat{u}_d$ is strictly concave and the minimum operator is concave, $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d))$ is strictly concave. Therefore, $\hat{\pi}$ is strictly concave as well. Thus, first-order conditions of optimality are sufficient for a local optimum. Similar to the proof of Theorem 2, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -q_d \check{a}_d + \left(1 - \beta_d - \gamma_{d'}\right) q_d \frac{\partial \check{a}_d}{\partial \beta_d} + \left(1 - \beta_{d'} - \gamma_d\right) q_{d'} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &+ \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned} \tag{B46}$$

In this region, from (B17), we have $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$. Because $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}_{d'}^d$ and $\frac{\partial \hat{u}_d}{\partial \beta_d} = a_d \hat{q}_d^d -$

$r\sigma^2(\beta_d+\rho\gamma_d)$, by applying the envelope theorem on $\hat{u}_d(\check{a}_d,\hat{q}^d)$:

$$\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\beta_d}=a_d\hat{q}_d^d-r\sigma^2(\beta_d+\rho\gamma_d)+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\beta_d}. \tag{B47}$$

Similarly, because $\frac{\partial\hat{u}_{d'}}{\partial\beta_d}=0$ and $\frac{\partial\hat{u}_{d'}}{\partial\check{a}_d}=\gamma_{d'}\hat{q}_d^{d'}$, from (B47) and (B26) we obtain

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(q_d-\hat{q}_d^d\right)+(1-\beta_d-\gamma_{d'})\,q_d\frac{3\check{a}_d}{8\beta_d}+(1-\beta_{d'}-\gamma_d)\,q_{d'}\frac{\check{a}_{d'}}{8\beta_d}\\ &\quad -r\sigma^2(\beta_d+\rho\gamma_d)+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\beta_d}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\beta_d}.\end{aligned} \tag{B48}$$

Consider now γ_d. We have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'}\check{a}_{d'}+(1-\beta_d-\gamma_{d'})\,q_d\frac{\partial\check{a}_d}{\partial\gamma_d}+(1-\beta_{d'}-\gamma_d)\,q_{d'}\frac{\partial\check{a}_{d'}}{\partial\gamma_d}\\ &\quad +\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(\check{a}_d,w_d))}{d\gamma_d}+\frac{d\hat{u}_{d'}(\check{a}_{d'},\hat{q}^{d'}(\check{a}_{d'},w_{d'}))}{d\gamma_d}.\end{aligned} \tag{B49}$$

Because $\frac{\partial\hat{u}_d}{\partial\gamma_d}=\check{a}_{d'}\hat{q}_{d'}^d$, $\frac{\partial\hat{u}_d}{\partial\check{a}_{d'}}=\gamma_d\hat{q}_{d'}^d$, and $\frac{\partial\check{a}_{d'}}{\partial\gamma_d}=\frac{\check{a}_{d'}}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_d(\check{a}_d,\hat{q}^d)$, we obtain that

$$\frac{d\hat{u}_d(\check{a}_d,\hat{q}^d(a_d,w_d))}{d\gamma_d}=a_{d'}\hat{q}_{d'}^d-r\sigma^2(\gamma_d+\rho\beta_d)+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\gamma_d}. \tag{B50}$$

Similarly, because $\frac{\partial\hat{u}_{d'}}{\partial\gamma_d}=0$, $\frac{\partial\hat{u}_{d'}}{\partial\check{a}_d}=\gamma_{d'}\hat{q}_d^{d'}$, and $\frac{\partial\check{a}_d}{\partial\gamma_d}=\frac{3\check{a}_d}{8\gamma_d}$, from (B50) and (B30) we obtain

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)+(1-\beta_d-\gamma_{d'})\,q_d\frac{3\check{a}_d}{8\gamma_d}+(1-\beta_{d'}-\gamma_d)\,q_{d'}\frac{\check{a}_{d'}}{8\gamma_d}\\ &\quad -r\sigma^2(\gamma_d+\rho\beta_d)+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\gamma_d}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\gamma_d}.\end{aligned} \tag{B51}$$

Thus, from (B48) and (B51), we obtain the first-order conditions

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d\left(q_d-\hat{q}_d^d\right)-r\sigma^2(\beta_d+\rho\gamma_d)+\frac{\Delta_d}{\beta_d}=0,\\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)-r\sigma^2(\gamma_d+\rho\beta_d)+\frac{\Delta_d}{\gamma_d}=0,\end{aligned} \tag{B52}$$

where $\Delta_d\equiv(1-\beta_d-\gamma_{d'})\,q_d\frac{3\check{a}_d}{8}+(1-\beta_{d'}-\gamma_d)\,q_{d'}\frac{\check{a}_{d'}}{8}+\gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8}+\gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(q_d-\hat{q}_d^d\right)+r\sigma^2\left(\beta_d^2+\rho\gamma_d\beta_d\right)=\gamma_d\check{a}_{d'}\left(q_{d'}-\hat{q}_{d'}^d\right)+r\sigma^2\left(\gamma_d^2+\rho\beta_d\gamma_d\right). \tag{B53}$$

By Lemma 1, we have that $\beta_d\check{a}_d\hat{q}_d^d=\gamma_d\check{a}_{d'}\hat{q}_{d'}^d$, which implies that

$$\beta_d\check{a}_dq_d+r\sigma^2\beta_d^2=\gamma_d\check{a}_{d'}q_{d'}+r\sigma^2\gamma_d^2 \tag{B54}$$

We will guess and verify that, due to the symmetry condition (S), it is optimal to implement symmetric effort, $\check{a}_d=\check{a}_{d'}=\check{a}$, and that $q_d=q$, $\eta_d=\eta$, and $Z_d=Z$. Define $f(x)\equiv x\check{a}q+r\sigma^2x^2$. Note $f'(x)=\check{a}q+2r\sigma^2x>0$ for $x>0$, so that f is monotonic over positive numbers and $f(\gamma_d)=f(\beta_d)$ if and only if $\gamma_d=\beta_d$. Thus, $\hat{q}_d^d=\hat{q}_{d'}^d=e^{-\frac{\eta}{2}}q$ and $\check{a}_d=e^{-\frac{\eta}{2}}Z\beta_d^{\frac{3}{4}}\beta_{d'}^{\frac{1}{4}}q$. In order to optimally implement the same effort, it must be that $\beta_d=\beta_{d'}$, so $\check{a}=e^{-\frac{\eta}{2}}Z\beta q$. Thus, we obtain the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d}=-Z\beta_d\hat{q}_d^d\left(q-\hat{q}_d^d\right)+(1-2\beta_d)\,q\hat{q}_d^d\frac{Z}{2}-r\sigma^2\beta_d(1+\rho)+\frac{Z\beta_d\left(\hat{q}_d^d\right)^2}{2}=0. \tag{B55}$$

Therefore

$$\beta_d^2\equiv\frac{1}{1+3\left(1-\hat{q}_d^d/q\right)+2r\sigma^2\left(1-|\rho|\right)/(Zq\hat{q}_d^d)}. \tag{B56}$$

After substitution, this gives HQ payoff

$$\hat{\pi}^2\equiv\frac{Z^2e^{-\eta}q^4}{Ze^{-\frac{\eta}{2}}q^2\left(4-3e^{-\frac{\eta}{2}}\right)+2r\sigma^2(1+\rho)}. \tag{B57}$$

Because β_d is the same for both divisions, this verifies that a is symmetric. Because HQ objective $\hat{\pi}$ is strictly concave on this region, there is only one solution on this region, which implies that the symmetric solution is the unique solution.

Case (C): Consider $H_d \in \left(-e^{\eta}, -e^{-\eta}\right)$ with $\beta_d > 0 > \gamma_d$. Following the same process as in case (B) above, we have

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -q_d \check{a}_d + (1-\beta_d - \gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \beta_d} + (1-\beta_{d'} - \gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned} \tag{B58}$$

Because in this region $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, from (B47) and (B26) we obtain that

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -a_d \left(q_d - \hat{q}_d^d\right) + (1-\beta_d - \gamma_{d'})\, q_d \frac{3\check{a}_d}{8\beta_d} + (1-\beta_{d'} - \gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8\beta_d} \\ &\quad - r\sigma^2 \left(\beta_d + \rho\gamma_d\right) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\beta_d}. \end{aligned} \tag{B59}$$

Consider now γ_d. We have that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -q_{d'} \check{a}_{d'} + (1-\beta_d - \gamma_{d'})\, q_d \frac{\partial \check{a}_d}{\partial \gamma_d} + (1-\beta_{d'} - \gamma_d)\, q_{d'} \frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}. \end{aligned} \tag{B60}$$

Because $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$ and $\frac{\partial \hat{u}_d}{\partial \check{a}_{d'}} = \gamma_d \hat{q}_{d'}^d$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} = a_{d'} \hat{q}_{d'}^d - r\sigma^2 \left(\gamma_d + \rho\beta_d\right) + \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8}. \tag{B61}$$

Similarly, because $\frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$, $\frac{\partial \hat{u}_{d'}}{\partial \check{a}_d} = \gamma_{d'} \hat{q}_d^{d'}$, and $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, by applying the envelope theorem on $\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'})$, we obtain that

$$\frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d} = \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \tag{B62}$$

Together (B61) and (B62) give that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) + (1-\beta_d - \gamma_{d'})\, q_d \frac{3\check{a}_d}{8\gamma_d} + (1-\beta_{d'} - \gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8\gamma_d} \\ &\quad - r\sigma^2 \left(\gamma_d + \rho\beta_d\right) + \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \end{aligned} \tag{B63}$$

Thus, from (B59) and (B63), we obtain the first-order conditions

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d \left(q_d - \hat{q}_d^d\right) - r\sigma^2 \left(\beta_d + \rho\gamma_d\right) + \frac{\Delta_d}{\beta_d} = 0, \\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) - r\sigma^2 \left(\gamma_d + \rho\beta_d\right) + \frac{\Delta_d}{\gamma_d} = 0, \end{aligned} \tag{B64}$$

where $\Delta_d \equiv (1-\beta_d - \gamma_{d'})\, q_d \frac{3\check{a}_d}{8} + (1-\beta_{d'} - \gamma_d)\, q_{d'} \frac{\check{a}_{d'}}{8} + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8}$, giving

$$\beta_d \check{a}_d \left(q_d - \hat{q}_d^d\right) + r\sigma^2 \left(\beta_d^2 + \rho\gamma_d\beta_d\right) = \gamma_d \check{a}_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) + r\sigma^2 \left(\gamma_d^2 + \rho\beta_d\gamma_d\right). \tag{B65}$$

Again, in this region, $\hat{q}_d^d = \left[e^{-\eta_d} \left|H_d\right|\right]^{\frac{1}{2}} q_d$, and $\hat{q}_{d'}^d = \left(2 - \left[e^{-\eta_d} \left|H_d\right|^{-1}\right]^{\frac{1}{2}}\right) q_{d'}$, where $H_d = \frac{\gamma_d a_{d'} q_{d'}}{\beta_d a_d q_d}$. Thus,

$$\gamma_d \check{a}_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = \gamma_d \check{a}_{d'} q_{d'} \left(e^{-\frac{\eta_d}{2}} \left|H_d\right|^{-\frac{1}{2}} - 1\right) = -\gamma_d \check{a}_{d'} q_{d'} - e^{-\frac{\eta_d}{2}} \left(\beta_d a_d q_d \left|\gamma_d\right| a_{d'} q_{d'}\right)^{\frac{1}{2}}. \tag{B66}$$

Similarly,

$$\beta_d \check{a}_d \hat{q}_d^d = e^{-\frac{\eta_d}{2}} \left(\beta_d \check{a}_d q_d \left|\gamma_d\right| \check{a}_{d'} q_{d'}\right)^{\frac{1}{2}} \tag{B67}$$

Therefore, after substitution, we obtain that (B65) becomes

$$\beta_d \check{a}_d q_d + r\sigma^2 \beta_d^2 = |\gamma_d| \check{a}_{d'} q_{d'} + r\sigma^2 \gamma_d^2. \tag{B68}$$

We guess again that HQ optimally implement the same effort from both divisions, $\check{a}_d = \check{a}_{d'}$, which implies that $f(|\gamma_d|) = f(\beta_d)$, where again $f(x) \equiv x\check{a}q + r\sigma^2 x^2$. This implies that $|\gamma_d| = \beta_d$, or equivalently, that $\gamma_d = -\beta_d$, so that $H_d = -1$. Thus, $\hat{q}_d^d = e^{-\frac{\eta}{2}} q$, and $\hat{q}_{d'}^d = \left(2 - e^{-\frac{\eta}{2}}\right) q$. To be consistent with this guess, it must be that $\beta_{d'} = \beta_d$, so that $\check{a}_d = \check{a}_{d'} = e^{-\frac{\eta}{2}} Z\beta_d q$. Substituting in $\hat{\pi}$ and differentiating we obtain

$$\frac{d\hat{\pi}}{d\beta_d} = -Z\beta_d \hat{q}_d^d \left(q_d - \hat{q}_d^d\right) - r\sigma^2 \beta (1+\rho) + \frac{1}{2}(1 - 2\beta_d) Zq\hat{q}_d^d + \frac{1}{2}\beta_d Z \left(\hat{q}_d^d\right)^2 \tag{B69}$$

$$\beta_d^3 \equiv \frac{1}{1 + 3(1 - 3\hat{q}_d^d/q) + 2r\sigma^2 (1-\rho)/(Zq\hat{q}_d^d)}. \tag{B70}$$

After substitution, this gives HQ payoff

$$\hat{\pi}^3 \equiv \frac{Z^2 e^{-\eta} q^4}{Ze^{-\frac{\eta}{2}} q^2 \left(4 - 3e^{-\frac{\eta}{2}}\right) + 2r\sigma^2 (1-\rho)}, \tag{B71}$$

which verifies the guess that HQ optimally implements symmetric effort. Comparing $\hat{\pi}^2$ and $\hat{\pi}^3$, observe that they differ only for the final term in the denominator. Thus, $\hat{\pi}^3 \gtreqless \hat{\pi}^2$ as $\rho \gtreqless 0$, and

$$\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} = \frac{Z^2 e^{-\eta} q^4}{Ze^{-\frac{\eta}{2}} q^2 \left(4 - 3e^{-\frac{\eta}{2}}\right) + 2r\sigma^2 (1-|\rho|)}. \tag{B72}$$

Case (D): If $\gamma_d > e^{\eta}\beta_d$, we have that $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = e^{-\eta} q_{d'}$, so

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'} q_{d'} \left(1 - e^{-\eta}\right) - r\sigma^2 (\rho\beta_d + \gamma_d) < 0, \tag{B73}$$

and setting $\gamma_d > e^{\eta}\beta_d$ is not optimal. Similarly, if $\gamma_d < -e^{\eta}\beta_d$, we have that $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = \left(2 - e^{-\eta}\right) q$

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = a_{d'} q_{d'} \left(1 - e^{-\eta}\right) + r\sigma^2 (|\gamma_d| - \rho\beta_d) > 0 \tag{B74}$$

and setting $\gamma_d < -e^{\eta}\beta_d$ is not optimal. Thus, under symmetry, $|H_d| \le e^{\eta}$.

The second and final step is to compare $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\}$ and $\hat{\pi}^1$. Let

$$f(\eta) \equiv 2\left(1 - e^{-\frac{\eta}{2}}\right)^2 Zq^2 + r\sigma^2 (1 - |\rho|) \left[e^{\eta}(1 + |\rho|) - 2\right], \tag{B75}$$

so that $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} > \hat{\pi}^1$ if and only if $f > 0$. Note $f(0) = -r\sigma^2 (1 - |\rho|)^2 < 0$,

$$f'(\eta) = 2\left(1 - e^{-\frac{\eta}{2}}\right) e^{-\frac{\eta}{2}} Zq^2 + r\sigma^2 e^{\eta} \left(1 - \rho^2\right) > 0 \tag{B76}$$

and $\lim_{\eta \to \infty} f(\eta) = +\infty$, which implies there is a unique $\bar{\eta}$ such that $\max\left\{\hat{\pi}^2, \hat{\pi}^3\right\} > \hat{\pi}^1$ if and only if $\eta > \bar{\eta}$. Thus, for $\eta \le \bar{\eta}$ the optimal contract is in Case (A), with $\beta_d = \beta_d^1$ and $\gamma_d = -\rho\beta_d$, leading to (23), and for $\eta > \bar{\eta}$ the optimal contract is in Case (B) for $\rho < 0$, with $\beta_d = \beta_d^2$ and $|\gamma_d| = \beta_d$, or in Case (C) for $\rho > 0$, with $\beta_d = \beta_d^3$ and $|\gamma_d| = \beta_d$, leading to (24).

Finally, note that the first term of f, $2\left(1 - e^{-\frac{\eta}{2}}\right)^2 Zq^2$, is strictly positive. Because $f(\bar{\eta}) = 0$, it must be that $r\sigma^2 (1 - |\rho|) \left[e^{\bar{\eta}} (1 + |\rho|) - 2\right] < 0$. This implies that $\frac{\partial f}{\partial r} = \sigma^2 (1 - |\rho|) \left[e^{\eta}(1 + |\rho|) - 2\right] < 0$ in a neighborhood of $\bar{\eta}$. By the implicit function theorem, we obtain that $\frac{d\bar{\eta}}{dr} = -\frac{\frac{\partial f}{\partial r}}{f'(\eta)} > 0$, and $\bar{\eta}$ is increasing in r. Finally, for $\rho \neq 0$, define $\eta_\rho \equiv -\ln(|\rho|)$ and note that

$$f(\eta_\rho) = 2\left(1 - \sqrt{|\rho|}\right)^2 Zq^2 + r\sigma^2 \frac{(1 - |\rho|)^2}{|\rho|} > 0 \tag{B77}$$

which implies that $\bar{\eta} < \eta_\rho$. ■

Proof of Corollary 1. In the proof of Theorem 3, we showed that $\beta_d a_d q_d + r\sigma^2 \beta_d^2 = |\gamma_d| a_{d'} q_{d'} + r\sigma^2 \gamma_d^2$. Define $f(\beta_d, |\gamma_d|) = \beta_d a_d q_d + r\sigma^2 \beta_d^2 - |\gamma_d| a_{d'} q_{d'} - r\sigma^2 \gamma_d^2$, and note that in an optimal contract, $f = 0$. Note also that

$f\left(\beta_d, \beta_d\right) = \beta_d \left(a_d q_d - a_{d'} q_{d'}\right) > 0$ and that

$$f\left(\beta_d, \frac{a_d q_d}{a_{d'} q_{d'}} \beta_d\right) = r\sigma^2 \beta_d^2 \left(1 - \frac{a_d^2 q_d^2}{a_{d'}^2 q_{d'}^2}\right) < 0. \tag{B78}$$

Thus, $f\left(\beta_d, |\gamma_d|\right) = 0$ implies $|\gamma_d| \in (\beta_d, \frac{a_d q_d}{a_{d'} q_{d'}} \beta_d)$ for $\frac{a_d q_d}{a_{d'} q_{d'}} > 1$, and $|\gamma_d| \in (\frac{a_d q_d}{a_{d'} q_{d'}} \beta_d, \beta_d,)$ for $\frac{a_d q_d}{a_{d'} q_{d'}} < 1$. ■

Proof of Theorem 4. We guess and verify that headquarters have positive exposure to both divisions, $\phi_d = 1 - \beta_d - \gamma_d > 0$, and that beliefs are as in case (ii) of Lemma 3, $H_d^{HQ} \in \left(e^{-\eta_{HQ}}, e^{\eta_{HQ}}\right)$. Because (19) binds and $r = 0$, HQ payoff $\hat{\pi}$ is equal to

$$\sum_{\substack{d,d' \in \{A,B\} \\ d \neq d'}} \left[a_d q_d - \left(1 - \beta_d - \gamma_{d'}\right) a_d \left(q_d - \hat{q}_d^{HQ}\right) - \beta_d a_d \left(q_d - \hat{q}_d^d\right) - \gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right)\right], \tag{B79}$$

where $\hat{q}^d = (\hat{q}_d^d, \hat{q}_{d'}^d)$ are division manager beliefs from Lemma 1, a_d are the Nash equilibrium effort levels from Lemma 2, and $\hat{q}^{HQ} = (\hat{q}_d^{HQ}, \hat{q}_{d'}^{HQ})$ are HQ beliefs from Lemma 3. The proof is in two steps and is similar to the proof of Theorem 2. First, we show that $\gamma_d < 0$ is suboptimal; then we find the optimal contract for $\gamma_d \geq 0$.

Similar to Theorem 2, switching from γ_d to $|\gamma_d|$ does not affect $\hat{q}_d^d$, and thus does not affect a_d and $\beta_d a_d \left(q_d - \hat{q}_{d'}^d\right)$. Letting again $\hat{q}_{d'}^{d+}$ be the belief held by a division manager when receiving $|\gamma_d|$ instead of $\gamma_d < 0$, we have that $\gamma_d a_{d'} \left(q_{d'} - \hat{q}_{d'}^d\right) = |\gamma_d| \, a_{d'} \left(q_{d'} - \hat{q}_{d'}^{d+}\right)$ for all $\gamma_d < 0$. This implies that

$$\left(1 - \beta_{d'} - |\gamma_d|\right) a_{d'} \left(q_{d'} - \hat{q}_{d'}^{HQ}\right) < \left(1 - \beta_{d'} - \gamma_d\right) a_{d'} \left(q_{d'} - \hat{q}_{d'}^{HQ}\right) \tag{B80}$$

for $\gamma_d < 0$ because $\hat{q}_{d'}^{HQ} < q_{d'}$, and thus that setting $\gamma_d < 0$ is dominated by offering its absolute value, $|\gamma_d|$.

Because HQ strictly prefers offering $|\gamma_d| > 0$ to all $\gamma_d < 0$, it is sufficient to consider $\gamma_d \geq 0$. If HQ sets $\gamma_d > e^{\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division manager beliefs are in case (i) of Lemma 1, with $\hat{q}_d^d = q_d$ and $\hat{q}_{d'}^d = e^{-\eta} q_{d'}$, giving $a_d = \beta_d Z_d q_d$. Thus, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'} \left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) < 0$ because $\hat{q}_{d'}^{HQ} \in \left(e^{-\eta_{HQ}} q_d, q_d\right)$ and $\eta_{HQ} < \eta$, so setting $\gamma_d > e^{\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ is not optimal. Similarly, if $0 < \gamma_d < e^{-\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$, division managers beliefs are in case (iii) of Lemma 1, with $\hat{q}_d^d = e^{-\eta_d} q_d$ and $\hat{q}_{d'}^d = q_d$, giving $a_d = \beta_d Z_d e^{-\eta_d} q_d$. In addition, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = a_{d'} \left(\hat{q}_{d'}^d - \hat{q}_{d'}^{HQ}\right) > 0$ because $\hat{q}_{d'}^{HQ} \in \left(e^{-\eta_{HQ}} q_d, q_d\right)$, so setting $\gamma_d < e^{-\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ is not optimal. This implies that HQ set $e^{-\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}} \leq \gamma_d \leq e^{\eta_d} \frac{\beta_d a_d q_d}{a_{d'} q_{d'}}$ and induce beliefs that are in case (ii) of Lemma 1, with $H_d \in \left(e^{-\eta}, e^{\eta}\right)$.

Similar to the proof of Theorem 2, we can express HQ's objective as

$$\hat{\pi} = \phi_A \check{a}_A \hat{q}_A^{HQ} + \phi_B \check{a}_B \hat{q}_B^{HQ} + \left(\hat{u}_A(a_A, \hat{q}^A(a_A, w_A)) - s_A\right) + \left(\hat{u}_B(a_B, \hat{q}^B(a_B, w_B)) - s_B\right), \tag{B81}$$

where $\phi_d = 1 - \beta_d - \gamma_{d'}$, $\hat{u}_d(\check{a}_d, \hat{q}^d) = \min_{\hat{q}^d \in K_d^{\hat{q}}} \hat{u}_d$, with $\hat{u}_d = s_d + \beta_d \check{a}_d \hat{q}_d^d + \gamma_d \check{a}_{d'} \hat{q}_{d'}^d - \frac{\check{a}_d^2}{2Z_d} = 0$, and $\check{a}_d$ is the Nash equilibrium of division managers given by (B17) in the proof of Lemma 2. Consider first

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} \quad = \quad & -\hat{q}_d^{HQ} \check{a}_d + \phi_d \check{a}_d \frac{\partial \hat{q}_d^{HQ}}{\partial \beta_d} + \phi_{d'} \check{a}_{d'} \frac{\partial \hat{q}_{d'}^{HQ}}{\partial \beta_d} + \phi_d \hat{q}_d^{HQ} \frac{\partial \check{a}_d}{\partial \beta_d} + \phi_{d'} \hat{q}_{d'}^{HQ} \frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ & + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \end{aligned} \tag{B82}$$

Because $\hat{q}^{HQ}$ solves (26), from the envelope theorem $\phi_d \check{a}_d \frac{\partial \hat{q}_d^{HQ}}{\partial \beta_d} + \phi_{d'} \check{a}_{d'} \frac{\partial \hat{q}_{d'}^{HQ}}{\partial \beta_d} = 0$, which, together with (B25) and (B26) from the proof of Theorem 2, gives

$$\frac{d\hat{\pi}}{d\beta_d} = -\check{a}_d \left(\hat{q}_d^{HQ} - \hat{q}_d^d\right) + \phi_d \hat{q}_d^{HQ} \frac{3a_d}{8\beta_d} + \phi_{d'} \hat{q}_{d'}^{HQ} \frac{a_{d'}}{8\beta_d} + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\beta_d}. \tag{B83}$$

Consider now γ_d. Applying again the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ}\right)$, we obtain

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} \quad = \quad & -\hat{q}_{d'}^{HQ} \check{a}_{d'} + \phi_d \hat{q}_d^{HQ} \frac{\partial \check{a}_d}{\partial \gamma_d} + \phi_{d'} \hat{q}_{d'}^{HQ} \frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ & + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}. \end{aligned} \tag{B84}$$

Substituting (B29) and (B30) from the proof of Theorem 2 gives

$$\frac{d\hat{\pi}}{d\gamma_d} = -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^{d}\right) + \phi_d \hat{q}_d^{HQ} \frac{3\check{a}_d}{8\gamma_d} + \phi_{d'} \hat{q}_{d'}^{HQ} \frac{\check{a}_{d'}}{8\gamma_d} + \gamma_d \hat{q}_{d'}^{d} \frac{\check{a}_{d'}}{8\gamma_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\gamma_d}. \tag{B85}$$

Thus, from (B83) and (B85) we obtain the first-order conditions

$$\frac{d\hat{\pi}}{d\beta_d} = -\check{a}_d\left(\hat{q}_d^{HQ} - \hat{q}_d^{d}\right) + \frac{\Delta_d}{\beta_d} = 0, \quad \frac{d\hat{\pi}}{d\gamma_d} = -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^{d}\right) + \frac{\Delta_d}{\gamma_d} = 0, \tag{B86}$$

where $\Delta_d \equiv \phi_d \hat{q}_d^{HQ} \frac{3\check{a}_d}{8} + \phi_{d'} \hat{q}_{d'}^{HQ} \frac{\check{a}_{d'}}{8} + \gamma_d \hat{q}_{d'}^{d} \frac{\check{a}_{d'}}{8} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8}$, giving

$$\beta_d \check{a}_d \left(\hat{q}_d^{HQ} - \hat{q}_d^{d}\right) = \gamma_{d'} \check{a}_{d'} \left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^{d}\right). \tag{B87}$$

Because, from Lemma 1, $\beta_d \check{a}_d \hat{q}_d^d = \gamma_d \check{a}_{d'} \hat{q}_{d'}^d$, we have that (B87) implies $\beta_d \check{a}_d \hat{q}_d^{HQ} = \gamma_d \check{a}_{d'} \hat{q}_{d'}^{HQ}$. Because $H_d^{HQ} \in \left(e^{-\eta_{HQ}}, e^{\eta_{HQ}}\right)$, from Lemma 3, $\phi_d a_d \hat{q}_d^{HQ} = \phi_{d'} a_{d'} \hat{q}_{d'}^{HQ}$. Thus, $\frac{a_{d'}\hat{q}_{d'}^{HQ}}{a_d \hat{q}_d^{HQ}} = \frac{\beta_d}{\gamma_d} = \frac{\phi_d}{\phi_{d'}}$. Define m_d such that $\beta_d = m_d \phi_d$, so $\gamma_d = m_d \phi_{d'}$, which implies $\phi_d = 1 - \beta_d - \gamma_{d'} = \frac{1}{1+m_d+m_{d'}}$, and thus $\beta_d = \gamma_d = \frac{m_d}{1+m_d+m_{d'}}$. Substituting in $\gamma_d = \beta_d$ into $\check{a}$ from Lemma 2, we have $\check{a}_d = \left(Z_d^3 Z_{d'}\right)^{\frac{1}{4}} e^{-\frac{\eta}{2}} (\beta_d^3 \beta_{d'})^{\frac{1}{4}} (q_d q_{d'})^{\frac{1}{2}}$. Substituting into HQ objective, we obtain

$$\hat{\pi} = (Z_A Z_B)^{\frac{1}{2}} q_A q_B (\beta_A \beta_B)^{\frac{1}{2}} \left[2e^{-\frac{\eta_{HQ}}{2}} e^{-\frac{\eta}{2}} \left(1 - \beta_A - \beta_B\right) + \frac{3}{2} e^{-\eta} (\beta_A + \beta_B)\right]. \tag{B88}$$

Differentiating, we obtain the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d} = (Z_A Z_B)^{\frac{1}{2}} q_d q_{d'} (\beta_d^{-1} \beta_{d'})^{\frac{1}{2}} \left[e^{-\frac{\eta_{HQ}}{2}} \left(1 - 3\beta_d - \beta_{d'}\right) e^{-\frac{\eta}{2}} + \frac{3}{4} e^{-\eta} (3\beta_d + \beta_{d'})\right] = 0, \tag{B89}$$

giving

$$e^{\frac{1}{2}(\eta - \eta_{HQ})} + 3\left(\frac{3}{4} - e^{\frac{1}{2}(\eta - \eta_{HQ})}\right)\beta_d + \left(\frac{3}{4} - e^{\frac{1}{2}(\eta - \eta_{HQ})}\right)\beta_{d'} = 0. \tag{B90}$$

Because this holds for both divisions, after solving we obtain

$$\beta_A = \beta_B = \frac{1}{4 - 3e^{\frac{1}{2}(\eta_{HQ} - \eta)}} = \frac{1}{1 + 3(1 - \hat{q}_d^d / \hat{q}_d^{HQ})} = \gamma_d, \tag{B91}$$

giving (31). Note $\beta < \frac{1}{2}$ because $\eta_{HQ} < \eta - 2\ln\frac{3}{2}$ and $H_d^{HQ} = \hat{H}_d \in \left(e^{-\eta_{HQ}}, e^{\eta_{HQ}}\right)$. This implies that $\check{a}_d = \frac{\left(Z_d^3 Z_{d'}\right)^{\frac{1}{4}} e^{-\frac{\eta}{2}} (q_d q_{d'})^{\frac{1}{2}}}{4 - 3e^{\frac{1}{2}(\eta_{HQ} - \eta)}}$, and thus that $\hat{q}_d^d = e^{-\frac{\eta}{2}} q_d \hat{H}_d^{\frac{1}{2}}$ and $\hat{q}_d^{HQ} = e^{-\frac{\eta_{HQ}}{2}} q_d \hat{H}_d^{\frac{1}{2}}$. Similarly, (29) and (30) follow by direct substitution. ■

Proof of Theorems 5-7. Because the participation constraint (19) binds, we can express HQ's payoff as

$$\hat{\pi} = \phi_A a_A \hat{q}_A^{HQ} + \phi_B a_B \hat{q}_B^{HQ} + \left(\hat{u}_A(a_A, \hat{q}^A(a_A, w_A)) - s_A\right) + \left(\hat{u}_B(a_B, \hat{q}^B(a_B, w_B)) - s_B\right), \tag{B92}$$

where $\phi_d = 1 - \beta_d - \gamma_{d'}$ and $\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = \min_{\hat{q}^d \in K_d^{\hat{q}}} \hat{u}_d$, with

$$\hat{u}_d(a_d, \hat{q}^d(a_d, w_d)) = s_d + \beta_d a_d \hat{q}_d^d + \gamma_d a_{d'} \hat{q}_{d'}^d - \frac{r\sigma^2}{2}\left(\beta_d^2 + 2\rho\beta_d\gamma_d + \gamma_d^2\right) - \frac{a_d^2}{2Z_d} = 0, \tag{B93}$$

where $\hat{q}^d$ is from Lemma 1, a_d is from Lemma 2, and $\hat{q}^{HQ}$ is from Lemma 3. Different from Theorem 4, and similar to Theorem 3, because of division manager risk aversion, HQ objective function π admits again multiple strict local maxima. The proof proceeds again in two steps. First, we consider candidate optimal contracts that induce division managers to hold one of four possible configurations of beliefs (implied by Lemma 1) in the same four cases examined in the proof of Theorem 3, Cases (A) to (D). Second, we compare payoffs to HQ from optimal contracts in these regions and we determine the globally optimal contract. We will show Case (D) is never optimal. Note that optimal contracts falling in Case (A) correspond to Theorem 5, Case (B) corresponds to Theorem 6, Case (C) corresponds to Theorem 7 part (i). Finally, the comparison of payoffs from Case (B) and Case (C) gives Theorem 7 part (ii).

Case (A): If $|H_d| < e^{-\eta_d}$, have $\hat{q}_d^d = e^{-\eta_d} q_d$ and $\hat{q}_{d'}^d = q_{d'}$, which do not depend on γ_d. Similarly, by Lemma 2,

$a_d = \beta_d Z_d e^{-\eta_d} q_d$, which implies that both a_d and $a_{d'}$ do not depend on γ_d. Therefore,

$$\frac{d\hat{\pi}}{d\gamma_d} = -\hat{q}_{d'}^{HQ} a_{d'} + \phi_d a_d \frac{\partial \hat{q}_d^{HQ}}{\partial \gamma_d} + \phi_{d'} a_{d'} \frac{\partial \hat{q}_{d'}^{HQ}}{\partial \gamma_d} + \phi_d \hat{q}_d^{HQ} \frac{\partial a_d}{\partial \gamma_d} + \phi_{d'} \hat{q}_{d'}^{HQ} \frac{\partial a_{d'}}{\partial \gamma_d} + \frac{d\hat{u}_d(a_d, \hat{q}^d(a_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(a_{d'}, \hat{q}^{d'}(a_{d'}, w_{d'}))}{d\gamma_d}, \tag{B94}$$

where, by the envelope theorem on $\hat{\pi}$, we have $\phi_d a_d \frac{\partial \hat{q}_d^{HQ}}{\partial \gamma_d} + \phi_{d'} a_{d'} \frac{\partial \hat{q}_{d'}^{HQ}}{\partial \gamma_d} = 0$. In addition, on this region, $\frac{\partial a_d}{\partial \gamma_d} = \frac{\partial a_{d'}}{\partial \gamma_d} = 0$, which implies that $\frac{d\hat{u}_d(a_d,\hat{q}^d(a_d,w_d))}{d\gamma_d} = \frac{\partial \hat{u}}{\partial \gamma_d} = a_{d'} \hat{q}_{d'}^d - r\sigma^2 (\rho \beta_d + \gamma_d)$ and $\frac{d\hat{u}_{d'}(a_{d'},\hat{q}^{d'}(a_{d'},w_{d'}))}{d\gamma_d} = \frac{\partial \hat{u}_{d'}}{\partial \gamma_d} = 0$. Thus,

$$\frac{\partial \hat{\pi}}{\partial \gamma_d} = a_{d'} \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) - r\sigma^2 (\rho \beta_d + \gamma_d). \tag{B95}$$

Because HQ has long exposure to the symmetric divisions, $\hat{q}_d^{HQ} = \hat{q}_{d'}^{HQ} = e^{-\frac{\eta_{HQ}}{2}} q$. Thus, $\frac{\partial \hat{\pi}}{\partial \gamma_d} = 0$ if and only if $\gamma = -M\beta$, where $M \equiv \rho - \bar{\rho}$ and $\bar{\rho} \equiv \frac{Z \hat{q}_d^d}{r\sigma^2} \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) = \frac{e^{-\eta} Z q^2}{r\sigma^2} \left(1 - e^{-\frac{\eta_{HQ}}{2}} \right)$. Following a similar approach, we obtain

$$\frac{d\hat{\pi}}{d\beta_d} = \hat{q}_d^d \hat{q}_d^{HQ} Z (1 - 2\beta_d) - M\beta_d \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d Z + \beta_d Z \left(\hat{q}_d^d \right)^2 - r\sigma^2 \beta_d (1 - \rho M). \tag{B96}$$

Note $1 - \rho M = 1 - \rho^2 + \rho\bar{\rho}$ and $1 - 2\rho M + M^2 = 1 - \rho^2 + \bar{\rho}^2$, so $1 - \rho M = 1 - 2\rho M + M^2 + \bar{\rho}(\rho - \bar{\rho})$. Also, $r\sigma^2 \beta_d \bar{\rho} (\rho - \bar{\rho}) = Z \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d (\rho - \bar{\rho})$. Thus, we obtain the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d} = \hat{q}_d^d \hat{q}_d^{HQ} Z (1 - 2\beta_d) + \beta_d Z \left(\hat{q}_d^d \right)^2 - 2M\beta_d \left(q_{d'} - \hat{q}_{d'}^{HQ} \right) \hat{q}_d^d Z - r\sigma^2 \beta_d \left(1 - 2\rho M + M^2 \right) = 0, \tag{B97}$$

which implies

$$\beta_d^4 \equiv \frac{1}{1 + 2(\rho - \bar{\rho}) \left(\frac{\hat{q}_{d'}^d}{\hat{q}_{d'}^{HQ}} - 1 \right) + \left(1 - \frac{\hat{q}_d^d}{\hat{q}_d^{HQ}} \right) + \frac{r\sigma^2 \left(1 - \rho^2 + \bar{\rho}^2 \right)}{Z \hat{q}_d^{HQ} \hat{q}_d^d}}, \tag{B98}$$

giving (32). After substitution, this gives HQ payoff

$$\hat{\pi}^4 \equiv \frac{e^{-(\eta_{HQ} + 2\eta)} Z^2 q^4}{\left(2M + 2(1 - M) e^{-\frac{\eta_{HQ}}{2}} - e^{-\eta} \right) e^{-\eta} Z q^2 + r\sigma^2 (1 - 2\rho M + M^2)}. \tag{B99}$$

Case (B): If $H_d \in (e^{-\eta}, e^{\eta})$, as in the proof of Theorem 4, applying the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ}\right)$, we have

$$\frac{d\hat{\pi}}{d\beta_d} = -\hat{q}_d^{HQ} \check{a}_d + (1 - \beta_d - \gamma_{d'}) \hat{q}_d^{HQ} \frac{\partial \check{a}_d}{\partial \beta_d} + (1 - \beta_{d'} - \gamma_d) \hat{q}_{d'}^{HQ} \frac{\partial \check{a}_{d'}}{\partial \beta_d} + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}. \tag{B100}$$

Because in this region $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, from (B47) and (B26), we have

$$\frac{d\hat{\pi}}{d\beta_d} = -a_d \left(\hat{q}_d^{HQ} - \hat{q}_d^d \right) + (1 - \beta_d - \gamma_{d'}) \hat{q}_d^{HQ} \frac{3\check{a}_d}{8\beta_d} + (1 - \beta_{d'} - \gamma_d) \hat{q}_{d'}^{HQ} \frac{\check{a}_{d'}}{8\beta_d} - r\sigma^2 (\beta_d + \rho\gamma_d) + \gamma_d \hat{q}_{d'}^d \frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'} \hat{q}_d^{d'} \frac{3\check{a}_d}{8\beta_d}. \tag{B101}$$

Consider now γ_d. Applying again the envelope theorem on $\hat{\pi}\left(\hat{q}^{HQ}\right)$, we have

$$\frac{d\hat{\pi}}{d\gamma_d} = -q_{d'} \check{a}_{d'} + (1 - \beta_d - \gamma_{d'}) \hat{q}_d^{HQ} \frac{\partial \check{a}_d}{\partial \gamma_d} + (1 - \beta_{d'} - \gamma_d) \hat{q}_{d'}^{HQ} \frac{\partial \check{a}_{d'}}{\partial \gamma_d} + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}. \tag{B102}$$

Because in this region $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$, from (B50) and (B30), we have that

$$\begin{aligned} \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'} \left(q_{d'} - \hat{q}^d_{d'} \right) + \left(1 - \beta_d - \gamma_{d'}\right) \hat{q}^{HQ}_d \frac{3\check{a}_d}{8\gamma_d} + \left(1 - \beta_{d'} - \gamma_d\right) \hat{q}^{HQ}_{d'} \frac{\check{a}_{d'}}{8\gamma_d} \\ & \quad -r\sigma^2 \left(\gamma_d + \rho\beta_d\right) + \gamma_d \hat{q}^d_{d'} \frac{\check{a}_{d'}}{8\gamma_d} + \gamma_{d'} \hat{q}^{d'}_d \frac{3\check{a}_d}{8\gamma_d}. \end{aligned} \tag{B103}$$

Thus, from (B101) and (B103) we obtain the first-order conditions

$$\begin{aligned} \frac{d\hat{\pi}}{d\beta_d} &= -a_d \left(\hat{q}^{HQ}_d - \hat{q}^d_d \right) - r\sigma^2 \left(\beta_d + \rho\gamma_d\right) + \frac{\Delta_d}{\beta_d} = 0 \\ \frac{d\hat{\pi}}{d\gamma_d} &= -a_{d'} \left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'} \right) - r\sigma^2 \left(\rho\beta_d + \gamma_d\right) + \frac{\Delta_d}{\gamma_d} = 0, \end{aligned} \tag{B104}$$

where $\Delta_d = \phi_d \hat{q}^{HQ}_d \frac{3a_d}{8} + \phi_{d'} \hat{q}^{HQ}_{d'} \frac{a_{d'}}{8} + \gamma_d \hat{q}^d_{d'} \frac{a_{d'}}{8} + \gamma_{d'} \hat{q}^{d'}_d \frac{3a_d}{8}$, giving

$$\beta_d a_d \left(\hat{q}^{HQ}_d - \hat{q}^d_d \right) + r\sigma^2 \left(\beta_d^2 + \rho\gamma_d\beta_d \right) = \gamma_d a_{d'} \left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'} \right) + r\sigma^2 \left(\rho\gamma_d\beta_d + \gamma_d^2 \right) \tag{B105}$$

From Lemma 1, we have $\beta_d a_d \hat{q}^d_d = \gamma_d a_{d'} \hat{q}^d_{d'}$. Also, because $\phi_d > 0$ and HQ has beliefs as in case (ii) of Lemma 3, with $\phi_d a_d \hat{q}^{HQ}_d = \phi_{d'} a_{d'} \hat{q}^{HQ}_{d'}$, we have

$$\beta_d a_d \hat{q}^{HQ}_d + r\sigma^2 \beta_d^2 = \gamma_d \frac{\phi_d}{\phi_{d'}} a_d \hat{q}^{HQ}_d + r\sigma^2 \gamma_d^2. \tag{B106}$$

We now show that $\phi_A = \phi_B$. Suppose to the contrary that $\phi_A > \phi_B$. Because (B106) holds for both divisions, $\beta_A > \gamma_A$ but $\beta_B < \gamma_B$. This would imply, however, that $\phi_A = 1 - \beta_A - \gamma_B < 1 - \beta_B - \gamma_A = \phi_B$, which is a contradiction. Similarly, $\phi_A < \phi_B$ would also imply a contradiction. Thus, $\phi_A = \phi_B$. Further, this implies

$$\left(a_d \hat{q}^{HQ}_d + r\sigma^2 \left(\beta_d + \gamma_d\right) \right) \left(\beta_d - \gamma_d\right) = 0. \tag{B107}$$

Since the first term is strictly positive, $\beta_d = \gamma_d$. Further, because the divisions are symmetric, the first-order conditions are symmetric, which implies the existence of a symmetric solution, $\beta_A = \beta_B$. Because the problem is strictly concave on this region, this must be the unique solution. Thus, $a_A = a_B = e^{-\frac{\eta}{2}} Z\beta q$. Also, $\hat{q}^{HQ}_d = \hat{q}^{HQ}_{d'} = e^{-\frac{\eta_{HQ}}{2}} q$ and $\hat{q}^d_d = \hat{q}^d_{d'} = e^{-\frac{\eta}{2}} q$, so $\Delta_d = (1 - 2\beta) e^{-\frac{\eta_{HQ}}{2}} q \frac{e^{-\frac{\eta}{2}} Z\beta q}{2} + \beta e^{-\frac{\eta}{2}} q \frac{e^{-\frac{\eta}{2}} Z\beta q}{2}$, which gives the first-order condition

$$\frac{d\hat{\pi}}{d\beta_d} = \frac{1}{2} Z \hat{q}^d_d \hat{q}^{HQ}_d - 2\beta Z \hat{q}^d_d \hat{q}^{HQ}_d + \frac{3}{2} Z\beta \left(\hat{q}^d_d\right)^2 - r\sigma^2 \beta \left(1 + \rho\right) = 0. \tag{B108}$$

and thus

$$\beta^5_d \equiv \frac{1}{1 + 3\left(1 - \hat{q}^d_d / \hat{q}^{HQ}_d\right) + \frac{2r\sigma^2(1+\rho)}{Z\hat{q}^{HQ}_d \hat{q}^d_d}} = \hat{\beta}, \tag{B109}$$

giving (33). After substitution, this gives HQ payoff

$$\hat{\pi}^5 \equiv \frac{Z^2 q^4 e^{-\left(\eta_{HQ} + \eta\right)}}{Zq^2 \left(4e^{-\frac{(\eta_{HQ}+\eta)}{2}} - 3e^{-\eta} \right) + 2r\sigma^2 \left(1 + \rho\right)}. \tag{B110}$$

Theorem 4 showed that $\gamma_d > 0$ is optimal when $r = 0$. Similarly, $\gamma_d > 0$ when $\rho = 0$. Further, for $\rho < 0$, granting $\gamma_d < 0$ results in a larger risk premium, $\frac{r\sigma^2}{2} \left(\beta_d^2 + 2\rho\beta_d + \gamma_d^2 \right)$, than setting $\gamma_d > 0$. Thus, Case (B) dominates Case (C) for all $\rho \leq 0$. To conclude the proof of Theorem 5, note that $\hat{\pi}^5 \geq \hat{\pi}^4$ if and only if $g_L \geq 0$, where

$$\begin{aligned} g_L &\equiv \left(2M + 2\left(1 - M\right) e^{-\frac{\eta_{HQ}}{2}} + 2e^{-\eta} - 4e^{-\frac{(\eta_{HQ}+\eta)}{2}} \right) e^{-\eta} Zq^2 \\ & \quad + r\sigma^2 \left(1 - 2\rho M + M^2 - 2e^{-\eta} \left(1 + \rho\right) \right). \end{aligned} \tag{B111}$$

and note that $g_L|_{\eta=\eta_{HQ}=0} = -r\sigma^2 \left(1 + \rho\right)^2 < 0$, which implies that $\hat{\pi}^4 > \hat{\pi}^5$ for $\eta = \eta_{HQ} = 0$. Note also that $\frac{\partial g_L}{\partial M} = 2\left(1 - e^{-\frac{\eta_{HQ}}{2}}\right) e^{-\eta} Zq^2 + 2r\sigma^2 \left(M - \rho\right) = 0$, because $M \equiv \rho - \bar{\rho}$ and $\bar{\rho} \equiv \frac{e^{-\eta} Zq^2}{r\sigma^2} \left(1 - e^{-\frac{\eta_{HQ}}{2}}\right)$, and thus that $\frac{\partial g_L}{\partial \eta} = -g_L + 2\left(e^{-\frac{(\eta_{HQ}+\eta)}{2}} - e^{-\eta} \right) e^{-\eta} Zq^2 + r\sigma^2 \left(1 - 2\rho M + M^2\right) > 0$ for all $g_L < 0$. This implies that, for a given η_{HQ}, there is a unique $\hat{\eta}$ so that $g_L\left(\hat{\eta}, \eta_{HQ}\right) = 0$, and for all $\eta > \hat{\eta}$, it is $g_L > 0$ and thus $\hat{\pi}^5 > \hat{\pi}^4$.

Consider now η_{HQ}. Note first that $\frac{\partial g_L}{\partial \eta_{HQ}} = \left(2e^{-\frac{\eta}{2}} - (1-M)\right) e^{-\frac{\eta_{HQ}}{2}} e^{-\eta} Zq^2 > 0$ for $\eta < \eta' \equiv -2\ln\frac{1}{2}(1-M)$. Substituting η' in g_L, we obtain

$$g_L|_{\eta=\eta'} \equiv \frac{(1+M)^2(1-M)^2}{8} Zq^2 + r\sigma^2\left(1 - 2\rho M + M^2 - \frac{(1-M)^2}{4}(1+\rho)\right) > 0, \tag{B112}$$

where the inequality is obtained by noting that $h(\rho) \equiv 1 - 2\rho M + M^2 - \frac{(1-M)^2}{4}(1+\rho)$ is linear in ρ for any given M, thus achieving its minimum at an endpoint. Because $h(1) = \frac{1}{2}(1-M)^2 > 0$ and $h(-1) = (1+M)^2 > 0$, we have that $h(\rho) > 0$ for all $\rho \in [-1,1]$, and thus that $g_L|_{\eta=\eta'} > 0$. This implies that in the neighborhood of $g_L = 0$, $\eta < \eta'$, and thus that $\frac{\partial g_L}{\partial \eta_{HQ}} > 0$. Thus, there is a unique $\hat{\eta}_{HQ}$ (allowing for the possibility that $\hat{\eta}_{HQ} = 0$) such that $\hat{\pi}^2 > \hat{\pi}^1$ for $\eta > \hat{\eta}_{HQ}$, proving Theorem 5.

Case (C): Consider $H_d \in \left(-e^{\eta}, -e^{-\eta}\right)$ with $\beta_d > 0 > \gamma_d$. This case gives part (i) of Theorem 7.

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\hat{q}_d^{HQ}\check{a}_d + (1-\beta_d-\gamma_{d'})\hat{q}_d^{HQ}\frac{\partial \check{a}_d}{\partial \beta_d} + (1-\beta_{d'}-\gamma_d)\hat{q}_{d'}^{HQ}\frac{\partial \check{a}_{d'}}{\partial \beta_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\beta_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\beta_d}.\end{aligned} \tag{B113}$$

Because $\frac{\partial \check{a}_d}{\partial \beta_d} = \frac{3\check{a}_d}{8\beta_d}$ and $\frac{\partial \check{a}_{d'}}{\partial \beta_d} = \frac{\check{a}_{d'}}{8\beta_d}$, from (B47) and (B26) we have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -a_d\left(\hat{q}_d^{HQ} - \hat{q}_d^d\right) + (1-\beta_d-\gamma_{d'})\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\beta_d} + (1-\beta_{d'}-\gamma_d)\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\beta_d} \\ &\quad -r\sigma^2(\beta_d + \rho\gamma_d) + \gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8\beta_d} + \gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\beta_d}.\end{aligned} \tag{B114}$$

Consider now γ_d. We have that

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\hat{q}_{d'}^{HQ}\check{a}_{d'} + (1-\beta_d-\gamma_{d'})\hat{q}_d^{HQ}\frac{\partial \check{a}_d}{\partial \gamma_d} + (1-\beta_{d'}-\gamma_d)\hat{q}_{d'}^{HQ}\frac{\partial \check{a}_{d'}}{\partial \gamma_d} \\ &\quad + \frac{d\hat{u}_d(\check{a}_d, \hat{q}^d(\check{a}_d, w_d))}{d\gamma_d} + \frac{d\hat{u}_{d'}(\check{a}_{d'}, \hat{q}^{d'}(\check{a}_{d'}, w_{d'}))}{d\gamma_d}.\end{aligned} \tag{B115}$$

Because $\frac{\partial \check{a}_d}{\partial \gamma_d} = \frac{3\check{a}_d}{8\gamma_d}$, $\frac{\partial \check{a}_{d'}}{\partial \gamma_d} = \frac{\check{a}_{d'}}{8\gamma_d}$, from (B61) and (B30) we obtain

$$\begin{aligned}\frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) + (1-\beta_d-\gamma_{d'})\hat{q}_d^{HQ}\frac{3\check{a}_d}{8\gamma_d} \\ &\quad + (1-\beta_{d'}-\gamma_d)\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8\gamma_d} - r\sigma^2(\gamma_d + \rho\beta_d) + \hat{q}_{d'}^d\frac{\check{a}_{d'}}{8} + \gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8\gamma_d}.\end{aligned} \tag{B116}$$

From (B114) and (B116) we obtain the first-order conditions

$$\begin{aligned}\frac{d\hat{\pi}}{d\beta_d} &= -\check{a}_d\left(\hat{q}_d^{HQ} - \hat{q}_d^d\right) - r\sigma^2(\beta_d + \rho\gamma_d) + \frac{\Delta_d}{\beta_d} = 0, \\ \frac{d\hat{\pi}}{d\gamma_d} &= -\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) - r\sigma^2(\gamma_d + \rho\beta_d) + \frac{\Delta_d}{\gamma_d} = 0,\end{aligned} \tag{B117}$$

where $\Delta_d \equiv (1-\beta_d-\gamma_{d'})\hat{q}_d^{HQ}\frac{3\check{a}_d}{8} + (1-\beta_{d'}-\gamma_d)\hat{q}_{d'}^{HQ}\frac{\check{a}_{d'}}{8} + \gamma_d\hat{q}_{d'}^d\frac{\check{a}_{d'}}{8} + \gamma_{d'}\hat{q}_d^{d'}\frac{3\check{a}_d}{8}$, giving

$$\beta_d\check{a}_d\left(\hat{q}_d^{HQ} - \hat{q}_d^d\right) + r\sigma^2\left(\beta_d^2 + \rho\gamma_d\beta_d\right) = \gamma_d\check{a}_{d'}\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) + r\sigma^2\left(\gamma_d^2 + \rho\beta_d\gamma_d\right). \tag{B118}$$

Because the first-order conditions are symmetric, there exists a symmetric solution: $\beta_A = \beta_B = \beta$ and $\gamma_A = \gamma_B = \gamma$. Thus, $a_d = a = e^{-\frac{\eta}{2}}Z\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}q$. This also implies that $\phi_A = \phi_B$, so $\hat{q}_d^{HQ} = e^{-\frac{\eta_{HQ}}{2}}q$. Also, $H_d = \frac{\gamma}{\beta}$, so $\hat{q}_d^d = e^{-\frac{\eta}{2}}\frac{|\gamma|^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}q$ and $\hat{q}_{d'}^d = (2 - e^{-\frac{\eta}{2}}\frac{\beta^{\frac{1}{2}}}{|\gamma|^{\frac{1}{2}}})q$. Thus, $\beta a\hat{q}_d^d = e^{-\frac{\eta}{2}}\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}aq$ and

$$\gamma a\left(\hat{q}_{d'}^{HQ} - \hat{q}_{d'}^d\right) = \gamma a e^{-\frac{\eta_{HQ}}{2}}q - 2\gamma aq - e^{-\frac{\eta}{2}}\beta^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}aq, \tag{B119}$$

which implies that

$$\beta a e^{-\frac{\eta_{HQ}}{2}}q + r\sigma^2\beta^2 = |\gamma|\left(2 - e^{-\frac{\eta_{HQ}}{2}}\right)aq + r\sigma^2\gamma^2 \tag{B120}$$

Because $\frac{\gamma}{\beta} \in \left(-e^{\eta}, -e^{-\eta}\right)$, there exists $\hat{\xi} \in \left(e^{-\eta}, e^{\eta}\right)$ such that $\gamma = -\hat{\xi}\beta$. Substituting in $a = e^{-\frac{\eta}{2}} Z\beta\hat{\xi}^{\frac{1}{2}} q$, (B120) is equivalent to $f\left(\hat{\xi}\right) = 0$, where

$$f\left(\hat{\xi}\right) \equiv \left[\left(2e^{\frac{\eta_{HQ}}{2}} - 1\right)\hat{\xi} - 1\right] e^{-\frac{\eta_{HQ}}{2}} e^{-\frac{\eta}{2}} \hat{\xi}^{\frac{1}{2}} Zq^2 + r\sigma^2 \left(\hat{\xi}^2 - 1\right) = 0. \tag{B121}$$

Note $f\left(e^{-\eta}\right) < 0 < f(1) = 2\left[e^{\frac{\eta_{HQ}}{2}} - 1\right] e^{-\frac{\eta_{HQ}+\eta}{2}} Zq^2$ and $f' > 0$, so $\hat{\xi} \in \left(e^{-\eta}, 1\right)$ for $\eta_{HQ} > 0$, but $\hat{\xi} = 1$ if $\eta_{HQ} = 0$. Comparative statics on $\hat{\xi}$ follow because $max\left\{\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \sigma^2}, \frac{\partial f}{\partial \eta}\right\} < 0 < \min\left\{\frac{\partial f}{\partial Z}, \frac{\partial f}{\partial q}, \frac{\partial f}{\partial \eta_{HQ}}\right\}$. Further, $\frac{\partial \hat{\pi}}{\partial \beta} = 0$ if and only if

$$\beta = \frac{1}{1 + \left(\frac{\hat{q}^d_{d'}}{\hat{q}^{HQ}_{d'}} - 1\right)\hat{\xi} + 2(1 - \frac{\hat{q}^d_d}{\hat{q}^{HQ}_d}) + \frac{2r\sigma^2\left(1-\rho\hat{\xi}\right)}{Z\hat{q}^{HQ}_d \hat{q}^d_d}}; \quad \gamma = -\hat{\xi}\beta < 0, \tag{B122}$$

giving (34). After substitution in $\hat{\pi}$, we have

$$\hat{\pi}^6 \equiv \frac{e^{-(\eta_{HQ}+\eta)}\hat{\xi} Z^2 q^4}{2e^{-\frac{\eta_{HQ}}{2}}\left[1 + \left(2e^{\frac{\eta_{HQ}}{2}} - 1\right)\hat{\xi}\right] e^{-\frac{\eta}{2}}\hat{\xi}^{\frac{1}{2}} Zq^2 - 3e^{-\eta}\hat{\xi}q^2 Z + r\sigma^2\left(1 - 2\rho\hat{\xi} + \hat{\xi}^2\right)}. \tag{B123}$$

Note $\hat{\pi}^6 \geq \hat{\pi}^4$ if and only if $g_S \geq 0$, where

$$\begin{aligned} g_S \quad \equiv \quad & \left(2M + 2(1-M)e^{-\frac{\eta_{HQ}}{2}} + 2e^{-\eta}\right) Zq^2 + e^{\eta} r\sigma^2 \left(1 - 2\rho M + M^2\right) \\ & -2e^{-\frac{\eta_{HQ}}{2}}\left[1 + \left(2e^{\frac{\eta_{HQ}}{2}} - 1\right)\hat{\xi}\right] e^{-\frac{\eta}{2}}\hat{\xi}^{-\frac{1}{2}} Zq^2 - r\sigma^2 \frac{\left(1 - 2\rho\hat{\xi} + \hat{\xi}^2\right)}{\hat{\xi}}, \end{aligned} \tag{B124}$$

with

$$\frac{\partial g_S}{\partial \eta} = e^{\eta} r\sigma^2 \left(1 - \rho^2 + (\rho - M)^2\right) + \{e^{-\frac{\eta_{HQ}}{2}} e^{-\frac{\eta}{2}}\left[1 + \left(2e^{\frac{\eta_{HQ}}{2}} - 1\right)\hat{\xi}\right]\hat{\xi}^{-\frac{1}{2}} - 2e^{-\eta}\} Zq^2. \tag{B125}$$

Note that $\left[1 + \left(2e^{\frac{\eta_{HQ}}{2}} - 1\right)\hat{\xi}\right]\hat{\xi}^{-\frac{1}{2}}$ is increasing and larger than 2 for $\hat{\xi} \in \left(e^{-\eta}, 1\right)$, so $\frac{\partial g_S}{\partial \eta} > 0$. Also, $\frac{\partial g}{\partial \eta_{HQ}} = -(1-M)e^{-\frac{\eta_{HQ}}{2}} Zq^2 + \left(1 - \hat{\xi}\right) e^{-\frac{\eta_{HQ}}{2}} e^{-\frac{\eta}{2}}\hat{\xi}^{-\frac{1}{2}} Zq^2$. Because $M < e^{-\eta} < \hat{\xi}$, we have that $\frac{\partial g}{\partial \eta_{HQ}} < 0$. Defining $\hat{\eta}$, $\hat{\eta}^{HQ}_1$ so that $g_S\left(\hat{\eta}, \hat{\eta}^{HQ}_1\right) = 0$, part (i) of Theorem 7 is proven.

Case (D): If $\gamma_d > e^{\eta_d}\beta_d$, $\frac{\partial a_d}{\partial \gamma_d} = 0$, so $\frac{\partial a_{d'}}{\partial \gamma_d} = 0$, and thus $\frac{\partial \hat{\pi}}{\partial \gamma_d} = -a_{d'}\left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'}\right) - r\sigma^2(\rho\beta + \gamma) < 0$, so $\gamma \leq e^{\eta}\beta$. Similarly, if $\gamma_d < -e^{\eta_d}\beta_d$, $\frac{\partial a_d}{\partial \gamma_d} = \frac{\partial a_{d'}}{\partial \gamma_d} = 0$, so $\frac{d\hat{\pi}}{d\gamma_d} = -a_{d'}\left(\hat{q}^{HQ}_{d'} - \hat{q}^d_{d'}\right) - r\sigma^2(\rho\beta_d + \gamma_d)$. Because $\phi_{d'} > 0 > \gamma_d$, $\hat{q}^{HQ}_{d'} < q_{d'} < \hat{q}^d_{d'}$. Also, $\rho \in (-1, 1)$. Thus, $\frac{d\hat{\pi}}{d\gamma_d} > 0$ for $\gamma_d < -e^{\eta_d}\beta_d$, so it must be that $\gamma_d \geq -e^{\eta_d}\beta_d$. Therefore, Case (D) is suboptimal.

All that remains to be shown is part (ii) of Theorem 7, by showing that $\hat{\pi}^5 \geq \hat{\pi}^6$ when η_{HQ} is large enough. Note $\hat{\pi}^5 \geq \hat{\pi}^6$ if and only if $g_E \geq 0$, where

$$\begin{aligned} g_E \quad \equiv \quad & 2e^{-\frac{\eta_{HQ}}{2}}\left[1 + \left(2e^{\frac{\eta_{HQ}}{2}} - 1\right)\hat{\xi}\right] e^{-\frac{\eta}{2}}\hat{\xi}^{-\frac{1}{2}} Zq^2 + r\sigma^2\left(1 - 2\rho\hat{\xi} + \hat{\xi}^2\right)/\hat{\xi} \\ & -4e^{-\frac{(\eta_{HQ}+\eta)}{2}} Zq^2 - 2r\sigma^2(1+\rho). \end{aligned} \tag{B126}$$

Note $\frac{\partial g_E}{\partial \hat{\xi}} = \frac{f(\hat{\xi})}{\hat{\xi}^2} = 0$. Note that

$$\frac{\partial g_E}{\partial \eta_{HQ}} = \left[-\left(1 - \hat{\xi}\right)\hat{\xi}^{-\frac{1}{2}} + 2\right] e^{-\frac{(\eta_{HQ}+\eta)}{2}} Zq^2 \geq 0 \tag{B127}$$

if and only if $\hat{\xi} \geq 3 - 2\sqrt{2}$. Recall $\hat{\xi}$ is strictly decreasing in η_{HQ}. This implies that g_E an inverse U-shaped function of η_{HQ} and that there is a unique η'_{HQ}, defined by $\hat{\xi}(\eta'_{HQ}) = 3 - 2\sqrt{2}$, such that $\frac{\partial g_E}{\partial \eta_{HQ}} > 0$ for $\eta_{HQ} < \eta'_{HQ}$ and $\frac{\partial g_E}{\partial \eta_{HQ}} < 0$ for $\eta_{HQ} > \eta'_{HQ}$. Next, we will show that $g_E > 0$ for all $\eta_{HQ} \geq \eta'_{HQ}$ and, thus, for all $\hat{\xi} \leq 3 - 2\sqrt{2}$. Note that, from (B121), we can express (B126) as

$$g_E = 4e^{-\frac{\eta_{HQ}}{2}} e^{-\frac{\eta}{2}}\left(\hat{\xi}^{-\frac{1}{2}} - 1\right) Zq^2 + \frac{f\left(\hat{\xi}\right)}{\hat{\xi}} + 2r\sigma^2\left[\frac{1}{\hat{\xi}} - 2\rho - 1\right]. \tag{B128}$$

The first term is positive because $\hat{\xi} < 1$, the second term is zero, and the third term is positive for all $\frac{1}{\hat{\xi}} > 3$, which is satisfied for $\hat{\xi} \leq 3 - 2\sqrt{2} < \frac{1}{3}$. This implies that $g_E(\eta_{HQ}) > 0$ for all $\eta_{HQ} \geq \eta^{'}_{HQ}$. Thus, if $g_E\,(0) \geq 0$, $g_E > 0$ for all $\eta_{HQ} > 0$, and thus define $\hat{\eta}_2^{HQ} \equiv 0$; otherwise, if $g_E\,(0) < 0$, there is a unique $\hat{\eta}_2^{HQ}$ such that $g_E(\hat{\eta}_2^{HQ}) = 0$, with $\hat{\eta}_2^{HQ} < \eta^{'}_{HQ}$, completing the proof of Theorem 7. ■

Proof of Corollary 2. Follows directly from equation (B107). ■

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