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# Does the Potential to Merge Reduce Competition?

Finance Working Paper N° 554/2018
February 2018

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## Abstract

We study anti-competitive mergers in a dynamic model with noisy collusion. At each instant, firms either privately choose output levels or merge, which trades off benefits of avoiding price wars against the costs of merging. There are three results. First, mergers are optimal when collusion fails (i.e., firms sufficiently deviate from a collusive regime). Second, long periods of collusion are likely, because colluding is dynamically stable. Therefore, mergers are rare. Third, mergers (and, in particular, lower merger costs) decrease pre-merger collusion, as punishments by price wars are weakened. Thus, although anti-competitive mergers harm competition ex-post, barriers and costs of merging due to regulation should be reduced to promote competition ex-ante.

Keywords: competition, imperfect information, industry structure, market power, mergers

JEL Classifications: D43, L12, L13, G34

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February 16, 2018

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# 1 Introduction

The market power doctrine states that the concentration of output among firms in an industry is a good measure of the industry's market power (Demsetz, 1973). The standard view is that anti-competitive mergers increase industry concentration and hence increase market power (Stigler, 1964), implying that anti-competitive mergers harm competition and hence need to be regulated.

This view fails to account for dynamics. Using a dynamic model of two tacitly colluding firms that have the potential to merge at some point in the future, we show that the potential to merge leads them to be more competitive than if there were no potential to merge. This is because the potential to merge reduces the punishments that firms can mete out to each other for the perception that they have deviated from a regime of tacit collusion.[1]

Our model builds on the work of Green and Porter (1984) and Abreu, Pearce, and Stacchetti (1986) in discrete time and Sannikov (2007) in continuous time. The firms share a market and choose output levels on an ongoing basis. The firms would like to collude but neither firm can observe the actions of the other firm. Instead, they observe price, which is influenced by both firms, but which also is influenced by the noise in demand. As a result, firms cannot directly infer the action of the rival firm, but instead must estimate it.

Our analysis departs from the above dynamic noisy collusion models in that the firms can merge in our model. Merging enables the firms to share information and coordinate outputs and thus to earn deterministic, higher profits. But because mergers are costly, they are optimal only when collusion fails sufficiently (and hence the risk of inefficient price wars is sufficiently high).

In a (benchmark) model without mergers, firms collude despite their inability to monitor output levels, because punishments, implemented via price wars, occur if lower prices can potentially arise from overproduction rather than noise. In equilibrium, price reductions arise due to noise only.

We then add mergers to examine their effect on equilibrium. Even when mergers are possible, firms continue to collude rather than merge to avoid incurring the merger cost. However, if a price war is imminent, they choose to merge when expected gains from merging compared to potential price wars exceed the fixed cost of merging.

[1]Cartels and collusion are common phenomena in practice (Miller, 2009; Harrington and Skrzypacz, 2011; Harrington, 2017), there is a trend toward higher concentration (Autor, Dorn, Katz, Patterson, and Van Reenen, 2017; De Loecker and Eeckhout, 2017; Grullon, Larkin, and Michaely, 2017; Gutiérrez and Philippon, 2017), and collusion appears especially important for merger regulation (Nocke and Whinston, 2010; Coate, 2012). Moreover, there are no efficiency gains from mergers in our model, so Williamson's (1968) trade-off between market power and efficiency gains is not present and does not explain our results.

The firm's continuation values satisfy a differential equation that determines an equilibrium manifold (or closed curve) in discounted value space.[2] We also derive dynamics of optimal output levels. There are two aspects of dynamic behavior that are important: the first is the local stability of regions of the equilibrium manifold, and the second is the strength of that stability, which corresponds to the expected movement toward or away from a region of the equilibrium manifold. The option to merge causes the collusive region of the equilibrium manifold to decrease in size. Collusion remains dynamically stable in the equilibrium with mergers; by contrast, the region of the equilibrium manifold close to the merger points is dynamically unstable. These two properties reinforce each other, so that the dynamic optimizing behavior of the firms actually delays the onset of mergers (i.e., makes them rarer qualitatively) with the result that the firms spend their pre-merger time in the collusive region. The rarity of mergers is consistent with the findings of Andrade, Mitchell, and Stafford (2001) that aggregate merger activity relative to market capitalization (or firms in the Center for Research in Security Prices database) is, on average, in the range of 1%–2% (or 2%–4%).[3]

The possibility of anti-competitive mergers (even if rare) does not mean, however, there is less competition. The potential for mergers increases competition, because mergers weaken potential punishments, so ex-ante mergers can be pro-competitive. Using observed mergers as an indication of heightened market power is not satisfactory, because mergers occur only when collusion fails sufficiently relative to the merger cost. Mergers may be the tip of an iceberg of collusion dynamics (especially industries with more homogeneous goods) that might not have been fully recognized. Thus, if the ex-ante, pro-competitive effect is large, regulators could facilitate mergers by lowering frictions, such as barriers, costs, and expenses formally or informally placed by merger regulation.[4]

We also contribute to the debate on regulators' shift in competitive effects theories from "coordinated effects" to "unilateral effects" cases.[5] Recent evidence, however, revives arguments that tacit collusion is a bigger concern than previously thought.[6] It suggests regulators – despite difficulties – should revert back to bringing "coordinated effects" cases. Yet, our results suggest the opposite.

[2] In our continuous-time approach, mergers are specified by a differential equation's boundary conditions that determine both the locus and shape of the equilibrium manifold. This determines the merger terms and the firms' pre-merger play when the firms may still tacitly collude. Sannikov and Skrzypacz (2007, 2010) use similar methods.

[3] Hoberg and Phillips (2010) find that mergers are less likely for firms that are more similar to rivals, which they call "competitive effect," because there are more firms with similar restructuring options so the potential to merge is less valuable. In our model, this corresponds to higher merger costs and hence greater potential for punishments.

[4] This complements the "static view" of merger regulation; e.g., https://www.ftc.gov/enforcement/merger-review.

[5] See Salinger (2005), Baker and Shapiro (2007), and U.S. Department of Justice and Federal Trade Commission, Horizontal Merger Guidelines (1992, revised 1997).

[6] Azar, Schmalz, and Tecu (2017) find common ownership in U.S. airlines generates tacit collusion far beyond regulatory standards for anti-competitive behavior and hence reinvigorate the need for "coordinated effects" cases.

We also offer insights into the interactions between product market dynamics and mergers, which is consistent with empirical patterns. First, the result that collusion is the predominant pre-merger condition in a dynamic model helps to explain the anomalous empirical finding that mergers often do not appear to increase market power: the pre-merger situation is one of collusion, not competition, and this collusion comes much closer to replicating monopoly output than it does to a pure duopoly outcome. Thus, when firms merge, little changes, but that is because they have already anticipated the increased market power from merging in dynamically optimal pre-merger outputs, that is, the net gain after accounting for the merger cost is zero. In particular, the change in competition at the time of the merger is determined by the merger cost. In particular, a higher fixed cost of merging implies a greater reduction in competition at the time of the merger, because firms optimally compete longer by moving closer to the price-war region if merger costs are higher.

Second, the optimality of mergers means firm values (or stock prices) gradually and smoothly incorporate optimal strategies to compete under noisy collusion or not to compete (i.e., to merge). As firms trade off the gains from avoiding fiercer product market competition against the gains from creating a monopoly net of fixed cost of merging, there are no announcement returns, which may not be surprising given the boundary conditions and lack of information revelation by mergers. Nevertheless, we note the model is consistent with weak support of the market power doctrine (Eckbo, 1983).

Third, interpreting the firm's continuation value as its stock price, the model has implications for pre-merger returns for acquirer and target. Defining the acquiring firm as the firm with the larger merger share, this firm experiences a run-up in stock price for some time prior to a merger. However, at some point closer to the merger boundary the more valuable (acquiring) firm starts experiencing negative pre-merger returns. At the same time, the less valuable (target) firm, which consistently loses value on the way out of the collusive market-sharing region, experiences positive returns prior to the announcement of the deal. These results are consistent with empirical evidence reported, e.g., by Schwert (2000), that returns to target shareholders around a takeover announcement are typically positive, whereas returns to bidding shareholders are typically negative.[7] In our model negative pre-merger returns to bidding shareholders can arise simply because of the competition dynamics of the noisy collusion equilibrium.[8]

[7] Asquith (1983) documented a similar switch in the direction of return growth near announcement dates.

[8] It has often been argued that when the market believes that a takeover is less likely to be value-enhancing, an acquiring firm's stock price declines just prior to the announcement of a deal. According to our dynamic model, this reasoning might be flawed or, at a minimum, misleading: the acquiring firm in our model experiences negative pre-merger returns simply because it loses its position as an incumbent relative to the less valuable position of the target

The rest of the paper is organized as follows. Section 2 discusses the related literature. Section 3 outlines the model, which is then solved in Section 4. Section 5 derives the model's implications. Section 6 concludes. The appendices contain derivations, proofs and other technical results.

## 2 Related literature

The merger literature is rich and diverse, with numerous theoretical and empirical papers. We review papers from both bodies of literature to provide context and contrasts with our model. Regarding the theoretical dimension, Gowrisankaran's (1999) paper builds a dynamic model of mergers in which firms are static Cournot competitors before merging, but are capacity-constrained and can achieve greater capacity via investment and also via merging; the firms solve a dynamic game in which capacities are chosen using a dynamic programming approach.[9]

Jovanovic and Braguinsky (2004) consider a model with a continuum of firms, which have managers who choose projects from a continuum. It sets up a market in which firms have good managers who seek to acquire firms with good projects (but bad managers, who are fired after the acquisition). It thus is about efficiency gains from acquisitions, and does not address the issue of market power associated with mergers, the focus of our paper.[10]

The paper by Nocke and Whinston (2010) constructs a theoretical model of the optimal dynamic regulation of mergers. Like the Jovanovic-Braguinsky paper, they set up a model with a collection of many firms, and in each period merger opportunities—in the sense that a merger can achieve market power, production efficiencies, or both—randomly occur, and the firms facing these opportunities must decide whether to apply for regulatory approval. The regulator must then decide whether to approve a merger, taking account of the direct consequences of the merger in the sense of improving competition, but also on the future incentives for firms to apply for approval. Firms have merger opportunities do not interact prior to this (specifically, they do not collude); thus, a firm's optimal decision problem is static, namely to propose a merger or not when the opportunity arises. Also, there is no assumption of information asymmetry—noise—and the resulting strategic behavior. Our paper is very different: the entire focus is on firm behavior in anticipation of a merger, and in the face of noise in a very strategic environment.

as market entrant, as it moves toward the merger boundary, i.e., entirely due to the dynamics of the equilibrium.

[9] Because the model focuses on static Cournot equilibria, firms do not install excess capacity as a punishment strategy; if such punishments were implemented, then from the perspective of our model the prospect of mergers would reduce collusion in capacity due to the weakening of punishments.

[10] Bernile and Lyandres (2015) find that operating efficiencies also influence announcement returns of mergers.

Bernile, Lyandres, and Zhdanov (2012) is also related to our paper. As Gowrisankaran, they assume there are two incumbents and a third firm that is a potential entrant. Also as in Gowrisankaran, the incumbents compete on price in an essentially static fashion, the dynamics arising from demand shocks and decisions to merge or enter. Using a real options approach in which demand shocks follow a continuous-time diffusion process, they assume that entry by the third firm has a fixed cost; increased demand from the demand shock process leads to an increased gain from merging and exercising market power from the merger, but this increases the attractiveness of entry by the entrant, which dilutes those profits. Thus, in their model there is an *indirect* fixed cost of merging—the dilution of profits arising from entry; in our model we assume an exogenous fixed cost of merging. They also characterize the merger threshold using value-matching and smooth-pasting conditions as we do. However because the competition between firms is essentially static, they do not model dynamic collusion that enables firms to enforce collusion via punishments, in the spirit of our model.

Finally, Hackbarth and Miao (2012) also bears similarities with Bernile, Lyandres, and Zhdanov (2012) and with our paper. They explore mergers in a continuous-time dynamic setting in which there are many firms. Demand evolves according to a diffusion process. A merger combines the firms' capital as well as changing the competitive environment. They characterize the equilibrium value of a firm in the post-merger environment, taking account of the reduced number of firms in that environment, and the resulting changed oligopoly equilibrium. They then view the potential to merge as a real option to be triggered when demand passes a threshold that justifies the fixed cost of merging. As with Bernile, Lyandres, and Zhdanov (2012), they assume full information, and also assume that the firms act myopically (i.e., optimize output as if in a static oligopoly), so that the firms do not collude dynamically.

Regarding the empirical literature, a number of articles conduct announcement return (event) studies. Eckbo (1983), Stillman (1983), and Eckbo and Wier (1985) initiated the study of the impact of mergers on market power by using capital market data. Their studies build on the proposition that, in an efficient market, any merger-induced change in future profits of firms competing in the same product market as the merging firms goes hand in hand with merger-induced abnormal announcement returns to merging and rival firms. The implicit assumption is that firms will incorporate the change in market power into their product market strategies only at the time of the merger announcement and hence the market will only then impound the effect of anti-competitive mergers into stock prices of the merging firms and their rival firms. These studies and

also more recent ones (see, e.g., Fee and Thomas (2004) and Shahrur (2005)) fail to find evidence of anti-competitive effects associated with horizontal mergers. It has been argued that this failure is attributable to the inability of the event study methodology to detect the impact of horizontal mergers on competition in the case of diversified firms (McAfee and Williams, 1988) or the deterrent effect of antitrust enforcement (Prager, 1992). In contrast to these papers, our paper shows that once the cost of merging is also taken into account, there should be no announcement returns due to anti-competitive effects of mergers. Moreover, the decline in competition at the time of a merger is greater for higher merger costs.

Ashenfelter, Hosken, and Weinberg (2015), summarized in Ashenfelter, Hosken, and Weinberg (2013a), find evidence for price increases due to the merger of two large U.S. beer brewing companies, Miller and Coors. These authors identify production efficiencies from the merger as stemming mainly from reduced trucking distances from the breweries and can thus extract this effect when measuring price increases stemming from the increased market power flowing from the merger. They argue that efficiency gains will result in *slow* price increases, and this is what they observe. They also make a case for market power price increases but these seem small. Our model also predicts that there will be price increases, but that they should be small.[11]

Ashenfelter, Hosken, and Weinberg (2013b) also bring empirical evidence to bear on mergers. This paper looks at the household appliance maker Maytag's acquisition by Whirlpool in 2005. This move was made in response to an attempted purchase by a Chinese firm. The main object of intuitive interest in the paper is Figure 3: using ovens as a benchmark, because ovens face a competitive market, they look at price increases of other appliances, such as dishwashers, dryers, and clothes washers where the merged firm had a greater market share, relative to ovens subsequent to the merger, but starting *at* the merger *announcement* in 2005. They find that for clothes washers there is a substantial price increase ($\approx$40%), but the price increase spans three years, taking effect gradually. It is thus difficult to discern whether this slow price increase is due to other effects; specifically, their study period straddles the 2008 financial crisis. Because their study period commences at the merger announcement date, the paper cannot compare pre-merger collusion and post-merger pricing.

[11] To be more precise, we predict that the change in joint valuation will be zero in the purest version of the model, but output prices will increase. This follows because the merger is a transition from duopoly to monopoly, resulting in aggregate output falling and output prices increasing, but the concomitant rise in discounted profit only just compensates for the cost of the merger.

# 3 Model

Our dynamic model with noisy collusion builds on Sannikov's (2007), which is applied to a duopoly. The key idea in our approach is that each firm behaves as a principal in an agency construct, treating the other firm as its agent, with the continuation-value of the agent as the state variable for the principal; correspondingly, each firm at the same time behaves as an agent reacting to what is effectively a contract set out by the other firm.[12]

The agency construction allows us to characterize the equilibrium via the firms' continuation values by mapping out a one-dimensional manifold, $\mathcal{E}$, in the plane defined by the continuation values, which comprises the largest equilibrium manifold. The equilibrium characterized by this manifold is Markovian, i.e., the movement along the manifold depends only on the current state of the continuation values.

This Markovian property means that the equilibrium state can be mapped one-to-one to the equilibrium vector of public information $(X^1, X^2)$ that is driven by the output decisions of the firms and by noise. We begin by treating this information as a state variable for the firms, and we then carry out a transformation or mapping of this state to the space of continuation values using stochastic calculus by way of our agency construction. This then maps out the equilibrium manifold $\mathcal{E}$ via Sannikov's main differential equation (Sannikov (2007), equation (24), p. 1309).

We add two elements to Sannikov's construction. The first is our treatment of "enforcement," that is, the restriction on the matrix of coefficients of the reduced-form noise processes that are the shocks to the continuation values along the equilibrium manifold. For a coherent equilibrium, the matrix of coefficients must be singular so that firms coordinate so as to move on the equilibrium manifold—this is what yields the "self-generating" property of the equilibrium manifold. The singularity property is essential if our refinement of the continuation values is to determine the manifold.

The second additional element concerns the boundary conditions. Given the dynamic and stochastic nature of the model, it is necessary to view the moment the firms merge as a stopping time, and the firms as choosing this as the optimal stopping time. Because *both* firms must choose the *same* optimal stopping time, there is a complication that goes beyond standard optimal stopping problems: the firms must somehow coordinate their stopping times.

As with ordinary optimal stopping problems, the coordination problem is solved by a smooth-

[12] Solving a dynamic contract via agents' continuation values as state variables is an established solution technique in the dynamic contracting literature; see Miao and Zhang (2015) for a summary of the literature.

pasting condition applied to each firm's agency problem. In the agency construct, both firms naturally choose the same smooth-pasting point, and hence the same stopping time, thus solving the simultaneity problem. But a transformation of the model must first be carried out to enable the smooth-pasting problem to be appropriately expressed.

We remark that our solution method—informational state variables transformed to a continuation-value state vector using stochastic calculus and an agency framework—has the potential to be applied to other versions of the continuous-time Abreu, Pearce and Stacchetti model. Sannikov's continuous-time approach made an intuitive, geometric interpretation of their model possible. Aside from modeling mergers, our approach makes further applications of their model tractable.

### 3.1 Actions

In every stage game, there are two firms that take private actions and see public signals, which depend on their actions. Brownian motion distorts public signals, so that firms face an imperfect monitoring problem with regard to each other's actions. Private action and public signals determine the firms' payoffs (i.e., discounted values in a dynamic game). The stage game is played continuously at each moment in time $t \in [0, \infty)$. That is, time is continuous, and uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $\{\mathcal{F}_t\}$.

We consider two firms that compete in an industry with differentiated products that are imperfect substitutes. That is, we have a duopoly with imperfectly observable actions where two firms continuously take private actions (outputs) $A_t = (A_t^1, A_t^2)$. That is, firm $i = 1, 2$ chooses an action $A_t^i \in \mathcal{A}_i \subseteq \Re_+$ for all $t \in [0, \ \infty)$.[13] The solution will follow from an ordinary differential equation that characterizes the boundary of the equilibrium set, $\mathcal{E}(r)$, i.e., the continuation value pairs achievable by all public perfect equilibria of the continuous-time game.

The firms observe the history of a vector of public signals (i.e., price increments) $dP_t$, which depend on their actions $A_t^1$ and $A_t^2$. The instantaneous prices of firms 1 and 2 before and after the merger are given by the levels (not increments) of the processes:

$$dP_t^1 \ = \ \left(\Pi_1 \ - \ \beta_1 A_t^1 \ - \ \delta_1 A_t^2\right) \, dt \ + \ \text{noise}\,, \tag{1}$$

and

$$dP_t^2 \ = \ \left(\Pi_2 \ - \ \delta_2 A_t^1 \ - \ \beta_2 A_t^2\right) \, dt \ + \ \text{noise}\,. \tag{2}$$

[13] In Sannikov (2007), firms' outputs are discrete. We allow them to range over a continuum for analytical tractability, which enables us to take derivatives rather than analyze the optimal strategy by comparing adjacent actions.

Because the products are imperfect substitutes the actions of both firms influence the prices of both outputs.

The instantaneous payoff functions are the product of output and price increments: $g_i(\cdot) = A^i\,dP^i$ for $i = 1, 2$. Using lowercase letters to denote choices of $A^i$ and realizations $dP^i$, we express the payoffs of the firms as:

$$g_1(a_1, a_2) \;=\; a_1\,dp^1 \;=\; a_1\left(\Pi_1 \;-\; \beta_1 a_1 \;-\; \delta_1 a_2\right), \tag{3}$$

and

$$g_2(a_1, a_2) \;=\; a_2\,dp^2 \;=\; a_2\left(\Pi_2 \;-\; \delta_2 a_1 \;-\; \beta_2 a_2\right). \tag{4}$$

Discounted profits are integrals of these instantaneous profits.

### 3.2 Information

Firms learn about each other's actions by observing prices; the linearity of the prices in equations (1)–(2) means that the information processes can be isolated from those observations and expressed as a continuous process with independent and identically distributed increments. Before the merger, firms do not see each other's actions, but they see a vector of signals $X_t$:

$$dX_t^1 \;=\; \frac{\left(A_t^1 + A_t^2\right)dP_t^2}{2\left(\Pi_2 \;-\; \delta_2 A_t^1 \;-\; \beta_2 A_t^2\right)} \;-\; \frac{\left(A_t^1 - A_t^2\right)dP_t^1}{2\left(\Pi_1 \;-\; \beta_1 A_t^1 \;-\; \delta_1 A_t^2\right)} \;=\; A_t^1\,dt \;+\; \sigma_1\,dZ_t^1\,, \tag{5}$$

and

$$dX_t^2 \;=\; \frac{\left(A_t^1 + A_t^2\right)dP_t^1}{2\left(\Pi_2 \;-\; \delta_2 A_t^1 \;-\; \beta_2 A_t^2\right)} \;-\; \frac{\left(A_t^2 - A_t^1\right)dP_t^2}{2\left(\Pi_1 \;-\; \beta_1 A_t^1 \;-\; \delta_1 A_t^2\right)} \;=\; A_t^2\,dt \;+\; \sigma_2\,dZ_t^2\,, \tag{6}$$

where $Z_t$ consists of two independent Brownian motions $Z_t^1$ and $Z_t^2$. The state space information vector $\Omega$ is thus characterized by all possible paths of $X_t$, and the public information filtration $\mathcal{F}_t$ is adapted to $X_t$.

It can make sense to consider equilibria that are measurable with respect to the history of public signals $(X_s)_{s\leq t}$, because this would then reflect the history of actions $(a_s)_{s\leq t}$ that are payoff-relevant; the current value of the public information $X_t$ is not payoff-relevant per se.

We go a step further and consider only equilibria in which $X_t$ is the state, which will map one-to-one onto the vector of value states. These equilibria include those that are characterized by the manifold that is the boundary of the set of all possible equilibria, and which is itself an equilibrium—the same manifold found by Sannikov in his example. In those equilibria, we derive an equivalence between $X_t$ and the vector of continuation-value states $W_t$.

Each firm faces the same vector of public information, $X_t$, as its state variable, but with only one control, namely its own output $a_{it}$; the other firm's output history, $(a_{-i,s})_{s\leq t}$ is taken as an exogenous stochastic process. We then map the continuation-value process of the rival firm into a single state variable for the firm; because both firms undergo this mapping, there is still effectively a *vector* of equilibrium states, $W_t$, that maps one-to-one with the informational state $X_t$.

### 3.3 Duopoly stage game

As in many other dynamic settings, there might be solutions to the repeated game that are simply the repetition of the Nash equilibrium of the stage game (see, e.g., Hackbarth and Miao (2012)). To focus on the cooperative equilibria, we follow Sannikov (2007) and rule out stage-game Nash equilibrium strategies. This in turn requires that we calculate those equilibria. The optimal full information Nash (duopoly) solutions are:

$$a_1^* \;=\; \frac{\delta_1\,\Pi_2 \;-\; 2\,\beta_2\,\Pi_1}{\delta_1\,\delta_2 \;-\; 4\,\beta_1\,\beta_2}\,, \tag{7}$$

and

$$a_2^* \;=\; \frac{\delta_2\,\Pi_1 \;-\; 2\,\beta_1\,\Pi_2}{\delta_1\,\delta_2 \;-\; 4\,\beta_1\,\beta_2}\,. \tag{8}$$

Substituting equations (7) and (8) into equations (3) and (4) yields the following flow payoff functions:

$$\pi_t^{d,1} \;=\; r\int_t^\infty e^{-r(s-t)} g_1\left(a_1^*, a_2^*\right) ds \;=\; \frac{\beta_1\left(\delta_1\,\Pi_2 \;-\; 2\,\beta_2\,\Pi_1\right)^2}{\left(\delta_1\,\delta_2 \;-\; 4\,\beta_1\,\beta_2\right)^2}\,, \tag{9}$$

and

$$\pi_t^{d,2} \;=\; r\int_t^\infty e^{-r(s-t)} g_2\left(a_1^*, a_2^*\right) ds \;=\; \frac{\beta_2(\delta_2\,\Pi_1 \;-\; 2\,\beta_1\,\Pi_2)^2}{\left(\delta_1\,\delta_2 \;-\; 4\,\beta_1\,\beta_2\right)^2}\,. \tag{10}$$

These are the payoffs from the only pure strategy Nash equilibrium in the set $\mathcal{A}^N$, which, as we show in the Appendix, needs to be ruled out in equation (B.13).

### 3.4 Monopoly stage game

When firms merge, they eliminate their respective information problems and share the monopoly profits available when they are able to act as a single firm via the merger. To define the gains from merging, we calculate the monopoly profit of the merged firm. This is straightforward because we assume that one of the consequences of the merger is that the noise is eliminated, and the firms obtain full-information monopoly profits.

After the merger, the full information monopoly solution applies:

$$a_1^\star \;=\; \frac{(\delta_1+\delta_2)\,\Pi_2 \;-\; 2\,\beta_2\,\Pi_1}{(\delta_1+\delta_2)^2 \;-\; 4\,\beta_1\,\beta_2}\,, \tag{11}$$

and

$$a_2^\star \;=\; \frac{(\delta_1+\delta_2)\,\Pi_1 \;-\; 2\,\beta_1\,\Pi_2}{(\delta_1+\delta_2)^2 \;-\; 4\,\beta_1\,\beta_2}\,. \tag{12}$$

This implies the following flow payoff function starting from any time $t$ after the merger, which is an exogenous or non-strategic outside option:

$$\pi_m \;=\; r\int_t^\infty e^{-r(s-t)}\sum_{i=1}^{2} g_i\left(a_1^\star, a_2^\star\right) ds \;=\; \frac{(\delta_1+\delta_2)\,\Pi_1\,\Pi_2 \;-\; \beta_1\,\Pi_2^2 \;-\; \beta_2\,\Pi_1^2}{(\delta_1+\delta_2)^2 \;-\; 4\,\beta_1\,\beta_2}\,, \tag{13}$$

i.e., it will be shared only between the firms – based on pre-merger equilibrium dynamics – when they decide to end the dynamic game by agreeing to the terms of a merger.

### 3.5 The sequence of play

A fundamental property of our model is that firms gain from delaying the merger. The merger entails fixed costs: these include substantial legal fees that are necessary to obtain regulatory approval, due diligence measures, the generation of asset valuations, investment bank fees, and so on. As a practical matter, costly post-merger physical changes can be necessary: when two airlines merge, one of the fleets will need to be repainted.

Pre-merger collusion delays incurring these costs. Thus, firms would be interested in agreeing to an interim period in which they collude, conditional on their discounted profits from collusion dominating the profits available from an immediate merger. However, because they have not yet merged, they cannot observe each other's actions, so they must play a noisy collusion game prior to merging.

The firms are *aware* of the game and understand that they will eventually merge, and play the pre-merger game with this understanding. This results in an equilibrium characterized by a one-dimensional manifold in the space of continuation values that the firms stochastically traverse until they merge. Characterizing this manifold will be the main analytical task.

To give a preview of this construction, we first note that the merger results in the transition to monopoly, but the firms need to agree on a division of the monopoly profits between the shareholders of the two firms when they merge. The various potential divisions of the profits can be expressed as a line in the space of the continuation values of the firms; we will refer to this line as the merger line. We make no assumption that there is a split to which they have already agreed, such as 50-50; rather, the split is itself endogenous to the equilibrium of the game, treating the merger line as a boundary.

The firms also need to agree on the duration of the collusion period before the merger; the merger will occur at some time that is a stopping time which must be the same for both firms, even though each firm must independently agree to merge at the same time as the other firm; we assume that each firm chooses an *optimal* stopping time, and that in equilibrium these stopping times coincide exactly.

Importantly, we make the following assumption about agreeing to merge: in addition to being able to decide on the output at each moment of the pre-merger collusion period, we assume that each firm can at any instant announce its willingness to merge at that moment, but the merger does not take place until and unless both firms simultaneously announce their willingness to merge; once this happens the merger occurs and is irreversible. So the firms announce their willingness to merge when it is *weakly* optimal to merge. Thus, we do not consider situations when the firms are willing to merge but can then back away from the merger.

Finally, we assume that the announcement of the willingness to merge is publicly observable; although output decisions are by construction noisy, the merger signal is not. This is key because a firm's *refusal* to merge at the appropriate moment would then open the way to an equilibrium punishment for refusal, should that be needed to sustain mergers in equilibrium. We discuss the possibility of alternative paths to the merger in Appendix G, showing that the needed punishments would not be renegotiation-proof and so cannot support equilibria.

### 3.6 The merger

We are now ready to conceptually discuss the merger. At the time of the merger, the joint continuation values reach a threshold at which the joint value of the dynamic noisy collusion game and the value of the full-information monopoly game minus the fixed cost of merging are the same. In addition, there must be no *marginal* benefit to each firm deviating from the continuation either before the merger or at the time of the merger.

The merger will take place at an equilibrium stopping time, $\mathcal{T}_m$, that is individually rational and optimal from the perspective of each firm. Because it is an *optimal* stopping time there really are *two* stopping times, ${\mathcal{T}_m}^1$, and ${\mathcal{T}_m}^2$, one for each firm; it is then a requirement of the equilibrium that they be equal:

$$ {\mathcal{T}_m}^1 = {\mathcal{T}_m}^2 = \mathcal{T}_m \,. \tag{14} $$

At the time of the merger, the firms form a monopoly and share the monopoly profits, possibly

asymmetrically. Firm $i$'s share is defined as:

$$\xi^i(X_{\mathcal{T}_m})(\pi_m - k)\,, \tag{15}$$

where $\pi_m$ is the (deterministic) payoff from entering the monopoly in equation (13), $\xi^i(\cdot)$ is firm $i$'s share in the monopoly as a function of the vector of states $X_{\mathcal{T}_m}$ at the time of the merger, with

$$\xi^1(X_{\mathcal{T}_m}) + \xi^2(X_{\mathcal{T}_m}) = 1\,, \tag{16}$$

and $k$ represents the fixed cost of merging. The sharing rule (15) says that the merger enables the two firms to shift away from dynamic duopoly payoffs to sharing monopoly payoffs net of the transaction costs of merging.[14]

At the time of the merger, the value of continuing without merging and the value of merging will be the same. Thus, firm $i$'s continuation value is given by

$$W^i(X^i_{\mathcal{T}_m}) \;=\; \xi^i(X^i_{\mathcal{T}_m})(\pi_m - k)\,. \tag{17}$$

Notice that we treat the stopping time as specific to each firm at this stage of the model construction. Even though we will ultimately demonstrate that a value-matching condition applies to the merger, it is conceptually possible for the merger to take place via a jump, that is, the discounted value of the firm just prior to the merger might differ from the value of the merger. However this cannot be the case in equilibrium play, because the firms anticipate this jump, and so the discounted value of the firm must approach the merged value as the stopping time approaches.

### 3.6.1 Pre-merger continuation value

Define the pre-merger continuation value $W^i_t(\cdot)$ as the mapping, $W^i : \mathcal{X} \to \Re_+$, from the state vector $X_t$ to firm $i$'s time $t$ payoff in the continuous-time game. Given the boundary condition (17), this yields for all $t < \mathcal{T}^i_m$ the following continuation value function for firm $i$:

$$W^i(X_t) \;=\; \sup_{\mathcal{T}^i_m, A^i(\cdot)} \mathbb{E}\left[ r\int_t^{\mathcal{T}^i_m} e^{-r(s-t)} g_i\left(A^1_s, A^2_s\right) ds + e^{-r(\mathcal{T}^i_m - t)}\xi^i(X_{\mathcal{T}^i_m})(\pi_m - k) \middle| \mathcal{F}_t \right], \tag{18}$$

taking as given the rival firm's action process $A^{-i}_t$, where $\mathbb{E}[\cdot|\cdot]$ denotes the conditional expectation operator at time $t$.[15] Note again that each firm chooses its own $\mathcal{T}^i_m$, and therefore it must be established that in equilibrium the two stopping times are equal, so condition (14) holds in equilibrium.

[14] In other words, the merger does not have any other synergies, such as better cost efficiency, economies of scale, efficient capital reallocation, etc.

[15] By taking the rival's action $A^2_t$ as given, firm 1 views $A^2_t$ as a drift function of the history of the signals $X_{it}$, expressed as the filtration $\mathcal{F}_t$, but firm 1 does not take account of its indirect influence on the rival's action via its effect on the signal. See Sannikov (2007), pp 1292-1293.

### 3.7 The formal problem and definition of equilibrium

The problem we consider is that of finding the maximal set of payoffs attainable in equilibrium in the repeated game between the two firms, subject to the constraint that players' continuation values can never fall below the merger line. This is because continuing to play the collusion equilibrium nets the firms more profit than merging due to the fixed cost of merging, insofar as every equilibrium point above the merger line Pareto-dominates at least part of the merger line.

At the merger line, the continuation values are by definition equal in the merger and no-merger states and so the merging does not affect the instantaneous outcome. However the *marginal* impact of merging must also be accounted for, and this is expressed as the requirement that the *shares* garnered by each firm at the moment of the merger must be locally optimal for each, conditional on the other firm's strategy. More formally, we can define the game as follows.

**Definition 1** *A Markov merger game is a repeated game as described on page 1292 of Sannikov (2007) with stage game that is a tuple* $\{N, (\mathcal{A}_i)_{i\in N}, (g_i)_{i\in N}\}$*, where* $N$ *is the set of players,* $(\mathcal{A}_i)_{i\in N}$*, is the action set of player* $i$*, and* $g_i$ *is the payoff of player* $i$*, and in addition a tuple*

$$\{\mathcal{T}_m^1, \mathcal{T}_m^2, \tilde{\xi}^1(\cdot,\cdot), \tilde{\xi}^2(\cdot,\cdot), S^1(\cdot),, S^2(\cdot)\},$$

*such that at any moment* $t$ *firms choose a publicly observable signal* $S_t^i$ *from the set* $\{$ *"do not merge," "agree to merge with share* $\tilde{\xi}(W_t^1, W_t^2)$*"*$\}$ *and where* $\mathcal{T}_m^1$ *and* $\mathcal{T}_m^2$ *are the stopping times defined as the first time the signal* $S^i =$ *"merge" is chosen by firm* $i$*, and such that if the firms agree to merge simultaneously, that is,* $\mathcal{T}_m^1 = \mathcal{T}_m^2 = \mathcal{T}_m$*, then the merger takes place and the sharing rule is implemented with sharing rules* $\tilde{\xi}^1(\cdot,\cdot)$ *and* $\tilde{\xi}^2(\cdot,\cdot)$*.*

Thus, a Markov merger game behaves just like Sannikov's game in the run-up to the merger, during which time the firms can be thought of as sending the "do not merge" signal (or at least one of them). At the moment of the merger, they both send the "merge" signal along with the choice of the sharing rule, and the merger takes place, and is irreversible.

We note as an aside that the game is expressed notationally in terms of the sharing rule $\tilde{\xi}$, which is a function of the continuation values $W_t^1$ and $W_t^2$, which differ from the sharing rule $\xi$ that was defined in equation (17), which is a function of the state vector $X_t$. The equivalence of $\xi$ and $\tilde{\xi}$ follows from the one-to-one nature of the mapping between the state vector $X_t$ and the vector of continuation values $W_t$.

We turn now to the definition of equilibrium.

**Definition 2** *A Markov merger game equilibrium consists of:*

- *an action profile* $(A^1, A^2)$ *and a sharing function pair* $(\tilde{\xi}^1, \xi^2)$ *that satisfy: the action profile* $(A^1, A^2)$ *is a PPE as defined in Sannikov (2007), that is, "A profile* $A = (A^1, A^2)$ *is a PPE if, for* $i \in 1, 2$ *the profile* $A^i$ *maximizes the expected discounted payoff of player* $i$ *given the strategy* $A^{-i}$ *of the opposing player after all public histories," where the payoff is defined in equation (18), and in addition,*
- *firms merge only if both firms simultaneously play "agree to merge" with shares defined by* $\tilde{\xi}(W^1, W^2)$*;*
- *the stopping times* $\mathcal{T}_m^1$ *and* $\mathcal{T}_m^2$ *are optimal for firm 1 and firm 2 respectively;*
- *the stopping times are identical, that is* $\mathcal{T}_m^1 = \mathcal{T}_m^2 = \mathcal{T}_m$*;*
- *the sharing rules are feasible, that is, they satisfy*

$$\tilde{\xi}^2(W^1(X_{\mathcal{T}_m}), W^2(X_{\mathcal{T}_m})) = 1 - \tilde{\xi}^1(W^1(X_{\mathcal{T}_m}), W^2(X_{\mathcal{T}_m})). \tag{19}$$

While the largest equilibrium set is driven by the following sharing rule along the merger line,

$$W^2 = (\pi_M - k) - W^1, \tag{20}$$

it is unclear how to ensure that, in equilibrium, the firms agree to merge *simultaneously*. Our solution procedure transforms the model so as to characterize optimality, as is commonly the case with optimal stopping problems, by a smooth-pasting condition at the moment of the potential merger that holds for both firms. It is straightforward to show that the smooth-pasting condition is satisfied simultaneously; this is how we then demonstrate simultaneity. Of course the smooth-pasting condition is a characteristic of the solution, so we provide a verification of its optimality.

Because there are two firms, it is not completely obvious how to specify the smooth-pasting condition. Much of the subsequent technical development will be devoted to transforming the optimization problem in equation (18), so that the smooth-pasting condition can be clearly expressed. Once this solution is in place, we will establish that the smooth-pasting condition ensures (local) incentive compatibility at the merger point. That is, satisfying the smooth-pasting condition means that neither firm wants to locally deviate from the merger equilibrium. We then analyze the solution numerically.

# 4 Solution

In this section we derive the solution to the dynamic game using stochastic calculus, in two stages. Our stochastic calculus approach complements Sannikov's (2007) geometric approach to deriving continuation values, firms' strategies, and the equilibrium manifold. Our approach is a practical recipe for solving such models, and facilitates a direct economic interpretation of the boundary conditions. More specifically, we can break down our solution procedure into the following steps:

Stage 1
- (i) Solve the conventional profit maximization problem for each firm, taking the other firm's action profile as given, using the public signal vector $X_t$ as the state vector;
- (ii) use the solution of the firm optimization problem to state the continuation-value process for each firm in terms of the state vector $X_t$;
- (iii) use the requirement of local promise-keeping or "enforcement," which necessitates a singular volatility matrix, to restrict the structure of the continuation-value processes;
- (iv) implicitly map the state vector $X_t$ into the continuation-value vector $W_t$ using calculus arguments, so that a firm's continuation value process is implicitly expressed as a function of the rival firm's continuation value, i.e., implicitly construct the mapping $\mathcal{M} : \mathcal{X} \to \mathcal{R}$.

Stage 2
- (i) Pose the profit maximization problem for each firm as an agency problem with the rival firm's continuation value as the state process;
- (ii) solve the principal's optimal stopping problem using value-matching and smooth-pasting conditions and verify the optimality of smooth pasting;
- (iii) verify equilibrium differential equation and also the simultaneity of merger decisions.

The public information vector $X_t$ is a two-dimensional that is influenced via the drift function by the actions of *both* firms. The end of our first stage of the solution procedure maps this two-dimensional state into a two-dimensional state of continuation values. We then consider the manifold defining the largest self-generating equilibrium set; this manifold is *one*-dimensional; from the perspective of each firm, its problem reduces to having a *one*-dimensional state vector, namely the continuation value of the other firm. This is what allows us to use the smooth-pasting condition as a way of characterizing the stopping time, and also to ensure that the optimal stopping times of the two firms are in fact the same stopping time. Also, the dimension of the state vector that characterizes the equilibrium is preserved, that is, $W_t$ and $X_t$ are both two-dimensional.

### 4.1 First stage: Bellman equations when the states are noisy signals of actions

Our solution strategy will be to posit that the public signal vector $X_t$ comprises the state vector for the firms' value functions. The processes $X_t^1$ and $X_t^2$ are states that are themselves functions of the histories, and that are controlled by the firms' actions. The *current* state $X_t$ can be attained via many paths of action (or histories), but only the current state matters. In our formulation, the state vector $X_t$ summarizes the *particular* histories that lead to the equilibrium in which we are interested, namely the largest equilibrium manifold, and $X_t$ fully characterizes these histories.

The required rate of return for the owners of firm 1 is the risk-free rate $r$. Thus, the Bellman equation for firm 1 in the continuation region of the continuous-time game is:

$$r\, W^1(X^1, X^2)dt \;=\; \max_{A^1} \Big\{ \mathbb{E}\left[ r\, g_1(A^1, A^2)\, dt + dW^1(X^1, X^2) \right] \Big\}, \tag{21}$$

where firm 1's maximization over $A^1$ suppresses the exclusion of its pure strategy Nash equilibrium in equation (7). Applying Ito's lemma to expand the right-hand side of the Bellman equation and dropping the $dt$ terms, it is easy to verify that, with the observed signals as states, firm 1's dynamic optimization problem is expressed in the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rW^1(X^1, X^2) \;=\; \max_{A^1} \Big\{ & r\, g_1(A^1, A^2) + A^1 W^1_{X^1}(X^1, X^2) + A^2 W^1_{X^2}(X^1, X^2) \\ & + \tfrac{1}{2}\sigma_1^2 W^1_{X^1 X^1}(X^1, X^2) + \tfrac{1}{2}\sigma_2^2 W^1_{X^2 X^2}(X^1, X^2) \Big\}, \end{aligned} \tag{22}$$

where the cross-partial terms have dropped out given that the noise terms are uncorrelated.

Invoking our assumption that the action space is a continuum, we can use a conventional derivative to generate the optimality condition:

$$r\, g_{1A^1} \;+\; W^1_{X^1}(X^1, X^2) \;=\; 0\,. \tag{23}$$

The envelope condition, that is, the substitution of the optimality condition (23) into the HJB equation (22), in combination with Ito's lemma generate the stochastic process of the value states of the firms. (The proof is omitted because it uses standard ideas contained in Appendix A).

**Lemma 1** *The firms' value states follow the processes*

$$dW^1 \;=\; r\left(W^1 - g_1\right) dt \;-\; \sigma_1 r g_{1A^1} dZ_t^1 \;+\; \sigma_2 W^1_{X^2} dZ_t^2\,. \tag{24}$$

*and*

$$dW^2 \;=\; r\left(W^2 - g_2\right) dt \;-\; \sigma_1 W^2_{X^1} dZ_t^1 \;+\; \sigma_2 r g_{2A^2} dZ_t^2\,. \tag{25}$$

Combining the two firms yields the vector process:

$$dW_t \;=\; r\,(W_t - g(A_t))dt \;+\; \begin{pmatrix} -\sigma_1 r g_{1A^1} & \sigma_2 W^1_{X^2} \\ \sigma_1 W^2_{X^1} & -\sigma_2 r g_{2A^2} \end{pmatrix} dZ_t\,, \tag{26}$$

where $r\,(W_t - g(A_t))$ is the drift of the process. Denoting the volatility matrix by $B$, we can write this more compactly as,

$$dW_t \;=\; r\,(W_t - g(A_t))dt \;+\; B_t\,dZ_t\,. \tag{27}$$

#### 4.1.1 Strategy for transforming the $X_t$ state to the $W_t$ state

Our next step is to transform this system by manipulating the partial derivatives $W_{X^2}$. Note that the marginal value of the firm's state with respect to the other firm's signal, $W^i_{X^2}$, ends up determining the volatility. This is what Sannikov calls *enforcement* of equilibrium actions. In the next step, we analyze the volatility matrix $B$; we have not yet pinned down the cross-coefficients in the volatility matrix, that is, the optimality-determined values for $W^1_{X^2}$ and $W^2_{X^1}$ in equation (27). We next turn to this element of the model. We use a simple strategy: the cross-partial derivatives $W^1_{X^2}$ and $W^2_{X^1}$ are slopes of the value-function vector $W$ with respect to the state vector $X$, but when we combine them we obtain slopes of $W^1$ in terms of $W^2$ and vice versa, and thus implicitly we obtain $W^1$ as a *function* of $W^2$. This is roughly analogous to finding the slope of an indifference curve by taking the ratio of the marginal utilities. This will mesh with our strategy of formulating the second-stage problem, in which we express $W^2$ as a stochastic process that is the *state* for $W^1$.

#### 4.1.2 Characterizing the volatility matrix with calculus arguments

The volatility matrix $B_t$ contains cross-partial derivatives that we can partially characterize.

**Proposition 1** *In a Markovian equilibrium, the volatility matrix is singular.*

**Proof:** The *enforcement* property must hold along the equilibrium path. This translates into the direction of movement of the Markovian state vector to being tangential to the equilibrium manifold. This in turn requires compressing the stochastic shocks into a single effective shock, with the direction of motion of that shock tangential to the manifold. This in turn requires a singular volatility matrix, such that the eigenvector of the subspace defined by the singular matrix is tangential to the manifold. The only detail is establishing that the tangency holds at each point of the manifold. But this is by construction: see equation (25) and later; the volatility matrix depends on the local state. ■

Because the volatility matrix is singular, the largest equilibrium set is a one-dimensional manifold in the space of continuation values with elements $(W^1, W^2)$. The slope of that manifold at any point is

$$\frac{dW^2}{dW^1} \tag{28}$$

This derivative can be expressed in two distinct ways using implicit differentiation:

$$\frac{dW^2}{dW^1} \;=\; \frac{W^2_{X^1}}{W^1_{X^1}} \;=\; \frac{W^2_{X^2}}{W^1_{X^2}} \tag{29}$$

Using the first equality, we can write:

$$W^2_{X^1} = W^2_{W^1} W^1_{X^1}\,. \tag{30}$$

Two of the derivatives, $W^2_{X^1}$ and $W^1_{X^2}$, appear in the volatility matrix $B$ in equation (27), and the other two appear in the optimality conditions (see, e.g., equation (23)). Therefore, we can write:

$$\frac{W^2_{X^1}}{-g_{1A^1}} \;=\; \frac{-g_{2A^2}}{W^1_{X^2}}\,. \tag{31}$$

Because the volatility matrix must be singular, any joint evolution of the noise processes is then forced to move the equilibrium point along this lower-dimensional manifold. Therefore we can also assert that the determinant of the volatility matrix must be zero, which means:

$$g_{1A^1} g_{2A^2} \;=\; W^1_{X^2} W^2_{X^1}\,, \tag{32}$$

which is equivalent to the previous equation.

### 4.2 Carrying out the transformation/mapping $\mathcal{M} : X_t \mapsto W_t$

We carry out now the second stage of the solution. We will write down a new optimization for firm 1 in which the continuation value $W^2$ is a state variable for firm 1. We therefore will need to state the partial derivatives of $W^1$ with respect to the state variable $W^2$ in the Bellman equation. Substituting from the optimality condition (23) into equation (30) results in the transformations,

$$W^2_{X^1} \;=\; -rW^2_{W^1} g_{1A^1} = -r\frac{1}{W^1_{W^2}} g_{1A^1}\,, \tag{33}$$

and, similarly,

$$W^1_{X^2} = -rW^1_{W^2} g_{2A^2}\,. \tag{34}$$

We will use these transformations to construct the second-stage problem.

We draw attention to the fact that equations (33) and (34) implicitly redefine the value functions $W^1$ and $W^2$ as functions of $W^2$ and $W^1$ respectively, and no longer as direct functions of the states $X^1$ and $X^2$. Rather than developing a new and separate notation for these functions, we retain the notation $W^i$ in the sequel, with the arguments of the functions evident from context.

### 4.3 The second-stage agency reformulation

We now re-solve the firms' optimization problems, this time with each firm treating the other as an agent. Because the equilibrium movement of the continuation values is now restricted to a one-dimensional manifold, it is possible to model each firm as having only *one* state variable along the manifold! In an explicit agency formulation, there would be an incentive constraint, which would simply express the requirement that the agent optimize in the face of the contract. But we have already solved that optimization problem, and so we treat the optimized continuation value simply as a state variable, with its dynamics already stated in equation (25). Once this is done we can express the problem as a standard optimal stopping problem that can be solved with a standard HJB equation and a smooth-pasting condition. We can then establish that the equilibrium consists of the agency solutions simultaneously and consistently holding for both firms. In this section we solve the optimization problem and in the next section we construct the equilibrium.[16]

#### 4.3.1 The state equation

To formulate the equivalent agency problem we first characterise the state variable process. Normalizing $\sigma_1^2 = \sigma_2^2 = 1$ and $\sigma_{12} = 0$, and eliminating $X_t^1$ as an argument by substituting from equation (33) into equation (25), the continuation value process for firm 2 in terms of $W_t^2$ is:

$$dW^2 \;=\; r\left(W^2 - g_2\right)dt \;-\; r g_{2A^2} dZ_t^2 \;+\; (-r)\,\frac{1}{W^1_{W^2}} g_{1A^1} dZ_t^1. \tag{35}$$

#### 4.3.2 The merger boundary from the agency perspective

Define the share function $\tilde{\xi}(\cdot,\cdot)$ as a function of the continuation values, noting that we use distinct notation to distinguish this function from the earlier function $\xi(\cdot,\cdot)$, as follows. First, we note that the two functions must be equivalent in the sense that

$$\tilde{\xi}(W^1(X^1,X^2), W^2(X^1,X^2)) = \xi(X^1,X^2)$$

[16] By using the already-solved value process to capture the agency approach, we are in essence using the "first-order" approach to agency—that is, using first-order conditions. But we need to show that this is valid later when we demonstrate that the smooth-pasting condition holds, and that as a consequence the second-order condition is also satisfied when each firm acts as the "agent."

For firm 1, the equation for its share of the net payoff from merging at time $t = \mathcal{T}_m$ is given by:

$$W^1_{\mathcal{T}_m}(W^2_{\mathcal{T}_m}) \;=\; (\pi_m - k) - W^2_{\mathcal{T}_m}(W^1_{\mathcal{T}_m}) \;=\; \xi^1(W^1_{\mathcal{T}_m}, W^2_{\mathcal{T}_m})(\pi_m - k). \tag{36}$$

Solving for the share of firm 1, $\tilde{\xi}^1$, yields:

$$\tilde{\xi}^1(W^1_{\mathcal{T}_m}, W^2_{\mathcal{T}_m}) \;=\; 1 - \frac{W^2_{\mathcal{T}_m}(W^1_{\mathcal{T}_m})}{\pi_m - k} \qquad \text{and therefore} \qquad \tilde{\xi}^1_{W^2_{\mathcal{T}_m}} \;=\; -\frac{1}{\pi_m - k} \tag{37}$$

#### 4.3.3 The agency contract objective

Observe that the objective for firm 1 in equation (18), which satisfies the value-matching condition (17), expresses the sharing rule in terms of the state vector, $(X^1_{\mathcal{T}_m}, X^2_{\mathcal{T}_m})$ at the stopping time $\mathcal{T}_m$. In the second-stage "agency" setup, with the state $W^2$ for firm 1 and vice versa, this becomes:

$$W^1_t\left(W^2_t\right) = \sup_{\mathcal{T}_m, A^1(\cdot)} \mathbb{E}\left[r\int_t^{\mathcal{T}_m} e^{-r(s-t)} g_1\left(A^1_s, A^2_s\right) ds + e^{-r(\mathcal{T}_m - t)}\tilde{\xi}^1(W^1_{\mathcal{T}_m}, W^2_{\mathcal{T}_m})(\pi_m - k)\big|\mathcal{F}_t\right], \tag{38}$$

taking as given the other firm's control process $A^2$.

We can separate the problem of finding the optimal action $A^1$ from the optimal stopping problem. That is, the HJB equation solves the optimal action problem and the smooth-pasting condition pins down the optimal stopping problem taking as given the optimal policy $A^1$.

The first step in this process is to write the Hamilton-Jacobi-Bellman optimization problem for firm 1, using $W^2$ as a state with state equation (35). The full statement of the HJB equation is then

$$\max_{A^1}\left\{r(g_1 - W^1) - r(g_2 - W^2)W^1_{W^2} + \tfrac{1}{2}\left(-rW^2_{W^1} g_{1A^1}\right)^2 W^1_{W^2W^2} + \tfrac{1}{2}\left(rg_{2A^2}\right)^2 W^1_{W^2W^2}\right\}. \tag{39}$$

with state process

$$dW^2 \;=\; r\left(W^2 - g_2\right)dt \;-\; rg_{2A^2}dZ^2_t \;+\; (-r)\,\frac{1}{W^1_{W^2}} g_{1A^1} dZ^1_t. \tag{40}$$

and with the boundary condition

$$W^1(W^2_{\mathcal{T}_m}) = \tilde{\xi}^1(W^1_{\mathcal{T}_m}, W^2_{\mathcal{T}_m})(\pi_m - k). \tag{41}$$

Thus, everything is expressed in terms of the state $W^2$. There is a similar expression for firm 2.

We summarize these steps by the following proposition:

**Proposition 2** *(i) The two-step procedure in which the optimal continuation values $W^i(X^1, X^2)$ as a function of the publicly observable signals reflects optimization with regard to the firm's action $a^i_t$ at time $t$, followed by the transformation to the agency problem, satisfies incentive compatibility, as defined in Sannikov (2007), p. 1298, Proposition 2.*

*(ii) The second-stage agency optimization satisfies promise-keeping as defined in Sannikov (2007), p. 1296, Proposition 1.*

**Proof:** (i) Incentive compatibility is equivalent to optimization in the face of the constraints imposed by the equilibrium. The optimization is explicit here, namely the use of the first-order condition to alter the continuation value; we just need to demonstrate that this is also equivalent to optimization relative to the second-stage state variables, $(W^1, W^2)$. But this follows from backward from the second stage to the first stage, and noting that each firm optimizes its Bellman equation as reflected in equations (22) (and the equivalent for firm 2) and the optimality conditions (23) (and the equivalent for firm 2).

(ii) Promise-keeping follows from examining the differential form of equation (5) of Sannikov (2007), p. 1296 (without public randomization); this is then our equation (24) and (25). ■

#### 4.3.4 Optimality: the smooth-pasting condition

There are three main conditions that comprise the solution to the optimization problem stated in (39)-(41). The first condition is the standard optimality condition: maximize (39) with respect to $A^1$. However, rather than calculating this derivative explicitly, we will convert the problem into an equivalent maximization problem (see equation (E.7)) and solve that problem instead.

The second condition is the value-matching condition: at the merger point, the continuation value is identical to the share of the monopoly value less the cost of merging:

$$W^1(W^2_{\mathcal{T}_m}) \;=\; \tilde{\xi}^1_{\mathcal{T}_m}(W^1_{\mathcal{T}_m}, W^2_{\mathcal{T}_m})(\pi_m - k)\,. \tag{42}$$

The third condition is the smooth-pasting condition. The smooth-pasting condition for firm 1 follows from differentiating the boundary function (37):

$$W^1_{W^2}(W^2_{\mathcal{T}_m}) \;=\; \tilde{\xi}^1_{W^2_{\mathcal{T}_m}}(\pi_m - k) \;=\; -1\,, \tag{43}$$

with a similar condition for firm 2.[17] We begin with a lemma about the smooth-pasting condition. We show that the smooth-pasting condition locally satisfies the second-order condition for the firm solving the agency problem.

**Lemma 2** *The smooth-pasting condition is necessary for a local optimum with respect to the action $A^1$ at the merger point.*

[17] We again draw attention to our notation: $W^2_{\mathcal{T}_m}$, viewed by firm 1 as a state variable, denotes firm 2's continuation value evaluated at the stopping time $\mathcal{T}_m$, whilst $W^1_{W^2}(W^2_{\mathcal{T}_m})$ denotes the partial derivative of firm 1's continuation value as a function of that state at the stopping time.

**Proof:** See Appendix E. ■

To recapitulate, we first solved each firm's optimization problem in the face of the other firm's fixed policy. One can interpret this as optimizing on the part of each firm in the face of a contract. This optimization yielded a value process for the firm. The rival firm also generates a continuation-value process. The first firm now solves a second-stage problem in which the rival firm's value process is a state variable; this is equivalent to the firm optimizing the contract and imposing an incentive compatibility constraint on the rival firm. That second-stage optimization results in a Bellman equation that can then be re-stated as above.

**Proposition 3** *The smooth-pasting condition implies equal stopping times:* $\mathcal{T}_m^1 = \mathcal{T}_m^2$.

**Proof:** This is a key result, as it underscores the usefulness of the agency two-stage solution procedure. The proof follows from two observations. First, for the value-matching condition to be met, that is, for the terminal point to be on the merger line, the value-matching condition is necessarily met for both firms simultaneously. Second, the smooth-pasting condition (in the stage 2 agency formulation) entails the condition

$$W^1_{W^2}(W^2_{\mathcal{T}_m}) = -1, \tag{44}$$

for firm 1; inverting the equation yields

$$W^2_{W^1}(W^1_{\mathcal{T}_m}) = -1, \tag{45}$$

which is the smooth-pasting condition for firm 1. Thus, satisfying the smooth-pasting formula for firm 1 necessarily satisfies the smooth-pasting formula for firm 2. ■

Although the smooth-pasting condition in equation (43) is straightforward to state once we have transformed the problem to agency form, it is worthwhile contemplating the economic meaning of the condition. As the firms are driven to the merger line by the realizations of the noise, they stay on the equilibrium manifold by trading current payoffs against future "promise-keeping" payoffs. Indeed, our agency reformulation makes this trade-off explicit, in the sense that each firm sees itself as a principal offering this trade-off to the other firm via an equilibrium "contract" that accounts for the fact that the other firm optimizes against this contract (i.e., it is incentive-compatible). We note that the smooth-pasting condition (43), which expresses incentive compatibility at the merger point, reflects—like Sannikov's (2007) incentive-compatibility condition (9) in what for us is the

pre-merger play—the trading of utility between the two firms. However, the rate of exchange is fixed by the slope of the merger line. We summarize with the following proposition:

**Proposition 4** *The action $A^i(\cdot)$ and $W^i(\cdot)$ that solve (39), (41), and smooth-pasting condition (43), taking as given $A^{-i} \in \mathcal{A}$, solve the optimal action and stopping problem (38).*

**Proof:** See Appendix E. ■

### 4.4 Equilibrium and characterization

To reach an equilibrium in the game, firms must choose optimal contracts in their role as principals, optimally reacting to the other firm's contract in their role as agents, and the contracts must be identical; furthermore, they must agree on an identical stopping time. The agency approach enables us to do this in a direct and tractable way. We begin with a lemma.

**Lemma 3** *For any manifold that is self-generating, satisfies the value-matching and smooth-pasting conditions, and which lies entirely above the merger line, it is not Pareto-improving to merge prior to reaching the merger line.*

**Proof:** Because the merger announcement is public, the firms could mutually agree to merge prior to attaining the merger line, that is, they could agree to jump to some point on the merger line prior to attaining it via evolution along the manifold. By hypothesis, the manifold lies above the merger line, so for at least one of the firms the jump to the merger would reduce its continuation value and it would be not be individually rational to agree to the merger. ■

We remark that it is possible to construct self-generating manifolds that satisfy the value-matching and smooth-pasting conditions, but which lie entirely *below* the merger line; these manifolds fail as equilibria precisely because it is Pareto-improving to jump to the merger line rather than evolve toward it via the self-generating manifold. We explore this in Appendix G.

**Proposition 5** *The value functions that solve (39)-(43) for both firms simultaneously constitute a Markov Merger Equilibrium.*

**Proof:** This is a direct consequence of satisfying Sannikov's main differential equation, equation (14) of Sannikov (2007), and the smooth-pasting condition (43). ■

We convert the optimization problem into the form it takes in equation (E.7) to facilitate the expression of the equilibrium manifold in terms of polar coordinates, which in turn makes the numerical solution of the model more straightforward. We turn to this agenda next.

#### 4.4.1 Converting the ODE into geometric form

The nonlinearity of the model forces us, as it did Sannikov, to solve the model numerically in the analysis of practical examples. Like Sannikov (2007), we adopt a reformulation of the second-stage optimized Bellman equations. We express the differential equation for the equilibrium manifold in polar coordinates to facilitate the computation of numerical solutions in the next section. The details of these derivations, which were not provided by Sannikov, are presented in the appendix.

The normal and tangent to the manifold are, respectively:

$$\mathbf{N}(\theta) = (\cos(\theta), \sin(\theta))\,, \qquad \text{and} \qquad \mathbf{T}(\theta) = (-\sin(\theta), \cos(\theta))\,. \tag{46}$$

As demonstrated in Appendix B, we can then express the value process in polar coordinate form. In particular, the curvature $\kappa(W)$ of the equilibrium manifold, which is determined by the transformed form of the optimized Bellman equations of the firms, is the derivative of the polar coordinate with respect to movement along the equilibrium manifold. For example, $\kappa$ would be zero if the manifold were locally a straight line. Thus, angle $\theta$ and arc length $\ell$, we have along the manifold that:

$$\frac{d\theta}{d\ell} = \kappa\,, \qquad \text{and} \qquad \frac{d\ell}{d\theta} = \frac{1}{\kappa}\,, \tag{47}$$

The left-hand side derivative can be expressed in terms of the derivatives of the continuation values using polar coordinates, which leads us to the differential equation we solve numerically:

$$\begin{pmatrix} \frac{dW^1(\theta)}{d\theta} \\ \frac{dW^2(\theta)}{d\theta} \end{pmatrix} = \begin{pmatrix} -\frac{\sin(\theta)}{\kappa(\theta)} \\ \frac{\cos(\theta)}{\kappa(\theta)} \end{pmatrix} = \frac{\mathbf{T}(\theta)}{\kappa(\theta)}\,. \tag{48}$$

We solve this ordinary differential equation numerically to determine the equilibrium manifold, $\mathcal{E}(r)$, which provides a benchmark equilibrium. If a merger occurs, we numerically solve for $\mathcal{E}(r)$ subject to the boundary conditions (i.e., value-matching and smooth-pasting in equations (17) and (43)), which are not present in Sannikov (2007). These boundary conditions for the merger will have non-trivial effects on the firms' pre-merger strategies and values, which we study in the next section.

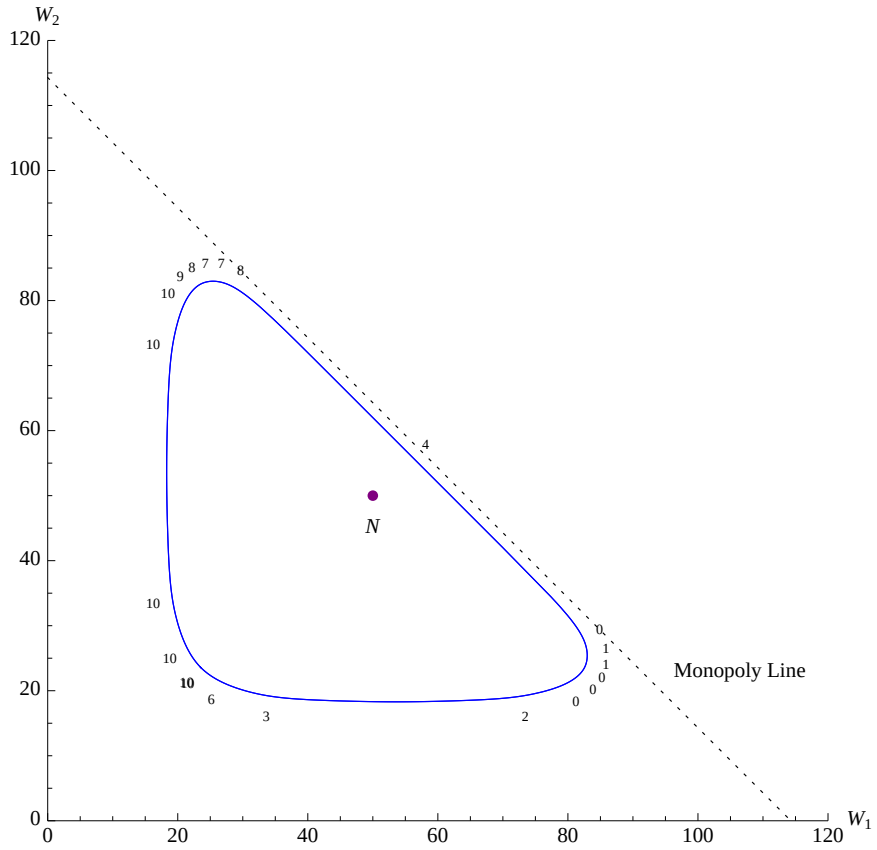
## 5 Implications

### 5.1 The no-merger equilibrium

To begin our analysis, we solve for the equilibrium of the benchmark model without mergers. The benchmark model's solution to the differential equation (48) is characterized by an equilibrium set, $\mathcal{E}(r)$, that forms a manifold in the space of continuation values, $(W^1, W^2)$, as seen in Figure 1,

where the continuation value of firm 1 (firm 2) is on the horizontal (vertical) axis. For the numerical analysis of the model's solution, we assume a baseline environment with symmetric demand functions and the following parameter values: $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.[18] Using equations (7) and (8), note that the static Nash equilibrium in the duopoly stage game is $(5, 5)$ in the baseline environment. This generates continuation values of $\pi_{d,i} = 50$ for each firm $i = 1, 2$ (or 100 for both firms).

**Figure 1. No-merger equilibrium manifold and outputs**




This figure plots the no-merger equilibrium manifold (blue, solid line). Firm 2's output choices are outside the equilibrium manifold. Note that, due to the symmetric demand functions, we can rotate firm 2's output choices around the 45 degree line to obtain firm 1's output choices. The static Nash equilibrium's output choices of (5,5) are depicted by $N$ in terms of the continuation values of $(50, 50)$. We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

Figure 1 displays the equilibrium manifold in the benchmark case without mergers along with

[18] Our symmetric example differs from Sannikov's (2007) asymmetric example, the parameter values of which are $\Pi_1 = 25$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 1$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1.5$. While the baseline parameter choices could be motivated in detail, we omit this for the sake of brevity and note that the model's results and implications vary quantitatively but not qualitatively with parameters.

firm 2's output choices. Due to the symmetric demand functions, we can rotate firm 2's output choices around the 45 degree line to obtain firm 1's output choices. Observe that the firms have a region, corresponding to the northeast stretch of the equilibrium manifold, in which they engage in cooperative behavior in that their output levels are around 4 and hence highly collusive. We refer to this as the *market-sharing* region. On the opposite side of the equilibrium manifold, that is, in the southwest stretch, they engage in a *price war*, in which output levels are around 10 and hence highly non-collusive. Put differently, the figure shows that total output increases as we move from the midpoint of the collusive region to the midpoint of the price-war region. Both firms have an incentive to overproduce because the winner of the overproduction-region gets rewarded by becoming dominant for some period of time in one of the two *contestability* regions that surround the price-war region. Subsequently, outputs move to (10,0) or (0,10), so one firm becomes dominant in output and reaches its maximal value on the equilibrium manifold, which is the *entrant and incumbent* region, before both firms potentially move to the market-sharing region or return to the contestability region.

In the upper right region of $\mathcal{E}(r)$ in Figure 1, output is low for both firms in that it sums up to about 8. That is, the firms approach the monopoly output, which is, according to equations (11) and (12), $a_i^\star = 3.75$ for each of the two firms (or 7.5 for both firms). Their continuation values are consequently higher in the dynamic duopoly's market-sharing equilibrium (e.g., around (56,56) at the midpoint of the market-sharing region) than in the static duopoly's Nash equilibrium, which corresponds in terms of continuation values to $(50, 50)$, depicted by $N$ in the figure. The monopoly value of the two firms is $\pi_m = 112.5$ and is depicted by the (dotted) *monopoly line* for all (feasible) sharing rules in the unit interval. Clearly, the monopoly value is unattainable in either the dynamic or the static duopoly. The northeast region is thus the Pareto frontier of $\mathcal{E}(r)$ where at least one firm colludes by producing less than the static best response and letting the other firm be the production leader. Recall that this the market-sharing regime. It is evident that, in this region, when a firm's continuation value increases, its market share also increases. Therefore, firms are tempted to overproduce by moving away from the center of the market-sharing region. In addition, the noisy observations of firms' actions make it difficult determine whether movement away from the market-sharing region is due to a series of random shocks (i.e., noise) or to less collusive output choices.

Observe that it is the upper left segment of the equilibrium manifold in Figure 1 where firm 2 obtains the maximal continuation value of almost 83, while the continuation value of firm 1 equals about 25. At that point, firm 1 underproduces, while firm 2 overproduces relative to the duopoly

and monopoly quantities. While firm 2 chooses actions closer to its (static) duopoly response, firm 1 needs strong incentives to "stay out." To reward firm 1 for staying out, firm 2 accommodates, and to punish firm 1 for deviating, firm 2 fights — the entrant and incumbent regime. Moreover, at the lower right and upper left, output is very asymmetric: at the lower right, for example, firm 1's output is high (i.e., 10) and firm 2's is low (i.e., 0). In the lower right segment of the equilibrium manifold, firms thus display similar strategies with the roles of firm 1 and 2 reversed, namely with firm 1 the incumbent and firm 2 as the entrant.

Finally, in the contestability and price-war regimes, output rises to 20, and continuation values decline for both firms. For example, above the center of the price-war region (i.e., intersection of $\mathcal{E}(r)$ with the 45 degree line), firm 1 is acting passively by producing close to its static best responses, while firm 2 aggressively overproduces. At this point, firm 2 is rewarded for overproducing by being able to drive firm 1 out of the market. This is the contestability regime, which will be the precursor of the *endogenous merger* regime in the next figure. At the intersection of $\mathcal{E}(r)$ with the 45 degree line, firms engage in a price war in that both firms aggressively overproduce. Their output levels are (10,10) and substantially exceed the (static) duopoly outputs of (5,5), which leads continuation values to drop well below (25,25). The firms have incentives to do so because the firm that looks more aggressive will come out as the winner of the price war (i.e., continuing to fight by overproducing while the other firm accommodates by underproducing). The winner of the price war gains in the contestability region, becomes the incumbent in the adjacent entrant and incumbent regime, and subsequently gains relatively more from the collusive play in the adjacent market-sharing regime.

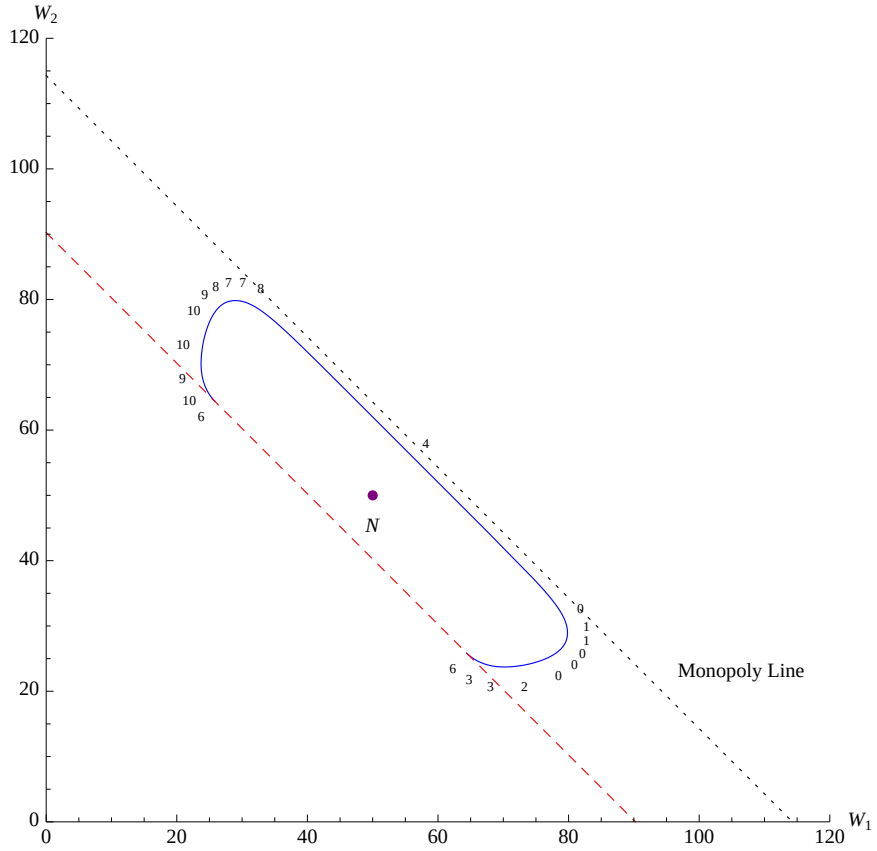
### 5.2 Merger equilibrium

We continue our analysis by solving the differential equation (48) for the equilibrium manifold, $\mathcal{E}(r)$, with mergers by incorporating the boundary conditions for value-matching and smooth-pasting in equations (17) and (43). In this solution of the dynamic game, firms also select dynamically optimal actions that are not static best responses. As a result, merger incentives arise endogenously when firms sufficiently deviate from their collusive actions. As illustrated in Figure 5, in the presence of an anti-competitive merger, the equilibrium manifold is significantly affected by the restructuring opportunity. When the merger occurs, both firms share (net of the merger cost) the value of the resulting monopoly stage game without imperfect information. This yields the *merger line* in equation (36), which corresponds to the monopoly line, $\pi_m$, minus the cost of merging, $k$, and is

represented by the red, dashed line in the figure for all (feasible) sharing rules in the unit interval. If firm 1, for example, captures more of the merger gains, then the merger point will more likely lie on the lower right section of the line; conversely, if firm 2 captures more of the gains, then the merger point is more likely to be on the upper left section of the line.

There are two potential states in which a merger can take place, and they apportion different shares to the firms.[19] The merger line smoothly pastes to two optimally determined merger points of the equilibrium manifold. We refer to the range between these two endpoints as the endogenous merger regime, which is, unlike the other regimes, an absorbing one. Therefore, the merger line presents an absorbing boundary for the equilibrium manifold.

**Figure 2. Merger equilibrium manifold and outputs**




This figure plots the merger equilibrium manifold (blue, solid line), the monopoly line (black, dotted line), and the merger line (red, dashed line) for a merger cost of $k = 24$. Firm 2's output choices are outside the equilibrium manifold. Note that, due to the symmetric demand functions, we can rotate firm 2's output choices around the 45 degree line to obtain firm 1's output choices. The static Nash equilibrium's output choices of (5,5) are depicted by $N$ in terms of the continuation values of $(50, 50)$. We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

[19] It is therefore more precise to state the stopping time $\mathcal{T}_m$ as $\mathcal{T}_{m_1} \wedge \mathcal{T}_{m_2}$, where $m_1$ and $m_2$ are the merger points.

Figure 2 establishes that firms' equilibrium behavior anticipates the impending merger, as seen both in terms of the dynamically optimal output choices and in terms of the resulting continuation values. In the equilibrium with mergers, the two firms are able to keep their outputs lower in a coordinated way than in the no-merger equilibrium. In particular, the equilibrium manifold with mergers is entirely contained inside the original no-merger equilibrium manifold in Figure 1. This means that some of the collusion profits attainable in the no-merger equilibrium are not attainable in the merger equilibrium, while some of the non-collusion costs (due to potential price wars) are avoided in the merger equilibrium. Intuitively, this stems from the weaker punishments inherent in the equilibrium with mergers. The punishments are weaker because the opportunity to merge eliminates the severe punishments in the price-war regime. The low payoffs from the price-war regime are "clipped away" by the endogenous mergers at the endpoints of the new (merger) equilibrium manifold. However, this weakens maximally possible punishments, which makes the entrant and incumbent region less attractive. In this sense, the dynamically optimal actions are more muted (i.e., involve less fighting and accommodating) in the merger equilibrium.[20]

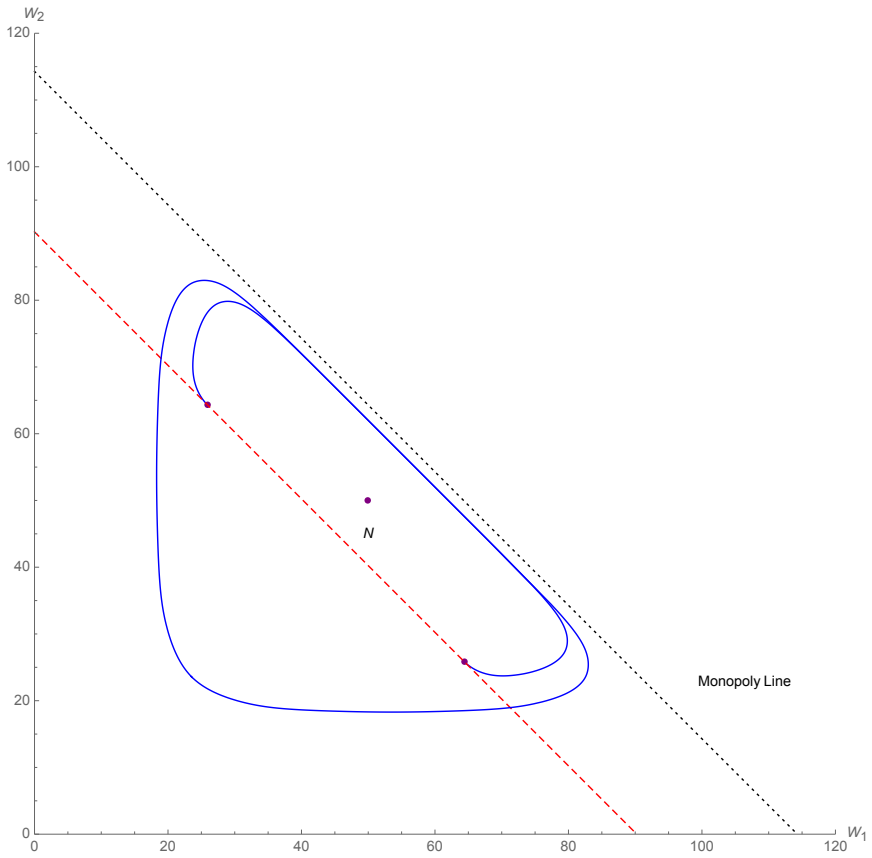
Observe that the (upper right) market-sharing region of the equilibrium manifold, which now reflects the possibility of a merger, is slightly less stretched out in Figure 2 compared with Figure 1. The firms trade off being the production leader in this noisy duopoly against being punished for deviating. As in the no-merger equilibrium, total output stays low at around 8, which is again very close to the optimal monopoly output of 7.5. In other words, the market-sharing regime, in which firms' optimal output levels are highly collusive, is similarly large relative to one without an anti-competitive merger, as in the previous figure. Moreover, as the entrant and incumbent regime is approached at the upper left region, total output increases to 10 and finally to 12 and 13 in the contestability region. But then total output declines slightly again to 12 just before the merger line is smoothly pasted to the equilibrium manifold. Thus, compared with the previous figure's no-merger equilibrium manifold, total output tends to be lower in the worst stages of the dynamic game. As a result of, on average, more collusive output choices (which generate, on average, higher continuation values in the presence of mergers) both firms are better off, even if they never merge. However, the maximally attainable continuation values in the entrant and incumbent regions are

[20] One might expect that as the cost of merging shrinks, the pure monopoly payoff manifold would be attained in the limit. However, this is not so: the weakening of punishments means that the maximum collusive profit shrinks as the merger cost shrinks. Moreover, below a threshold minimum but strictly positive cost (in our example around 8.1), the merger equilibrium no longer exists; we conjecture that firms simply merge immediately below this threshold.

slightly lower (around 80 instead of 83), because the threat of punishments is weaker in the presence of mergers. In sum, allowing for anti-competitive mergers implies pronounced periods of pre-merger collusion and leads to less overproduction (i.e., attenuated behavior in the entrant and incumbent regime and potentially complete avoidance of the price-war region).

Figure 3 illustrates the pro-competitive effect of mergers. The merger equilibrium manifold lies entirely inside the no-merger equilibrium manifold, and in the collusive region—the northeast part of the equilibrium manifold—the merger manifold lies to the southeast of the no-merger manifold, although the distinct difference between the manifolds is not visible in the figure. As this region is highly stable, this figure demonstrates that collusion is weakened by the potential to merge.

**Figure 3. Merger and no-merger equilibrium manifolds**




This figure plots the merger equilibrium manifold from Figure 2 and the no-merger equilibrium manifold from Figure 1. The merger equilibrium manifold is contained entirely within the no-merger equilibrium manifold. The value in the collusive region of the merger manifold is therefore below the value in the collusive region of the no-merger manifold, expressing the reduced punishments and reduced competition of the merger manifold.

In practice, merger gains (or payoffs) are also typically split asymmetrically between the merg-

ing firms. The firm that is being punished in the contestability region gets a smaller share of the merged entity's value, because it has a smaller continuation value and hence it appears to be taken over by the overproducing firm that has a larger continuation value in the contestability region. (We will therefore refer to the former as the target and the latter as the acquirer when we analyze merger returns.) This provides an explanation for asymmetric equity shares based on the product market's regime in the dynamic game (i.e., output strategies). The firm that overproduces at the right time will be rewarded by the larger share in the merged entity if the merger boundary is reached. Notably, this asymmetry is not driven by any inherent asymmetry in the demand functions, noise parameters, or other parameters, which we have assumed away precisely to make this point. The asymmetry in equity shares of the merged entity is therefore driven, in general, by dynamic optimization and, in particular, by the state of product market competition that the firms have attained as a result of cumulative play of the noisy duopoly game.

### 5.3 Collusion and the dearth of mergers

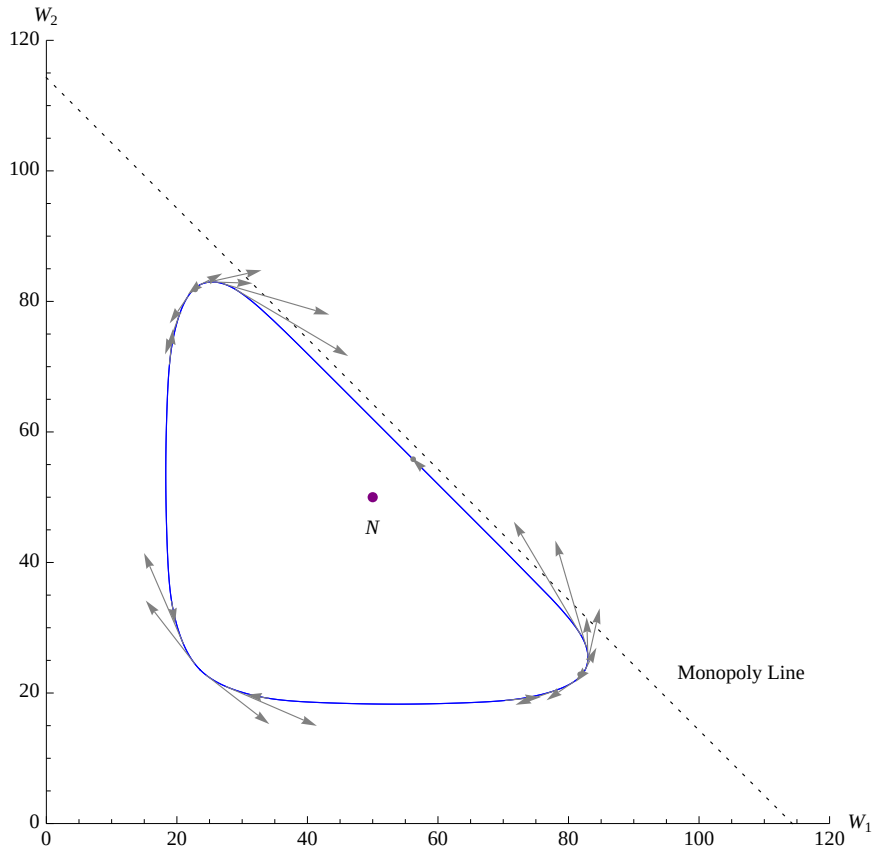
As a next step in the analysis, we examine the stability of the merger and the no-merger equilibria.[21] There are two aspects of dynamic behavior that are important: the first is the local stability of regions of the equilibrium manifold, and the second is the strength of that stability, which corresponds to the expected movement away from a region of the equilibrium manifold. The arrows in Figures 4 and 5 provide information about the stability of the regions. The length of the arrows, which corresponds to the volatility-scaled drift of the continuation value process represented in equation (27), indicates the strength of stability. The direction of the arrows conveys information about the local stability of each region.

Given that the arrows point away from each other in the price-war region of Figure 4, it is unstable. On the other hand, the arrows point towards each other in the collusive market-sharing region, indicating that it is stable. Even though the collusive region is smaller in Figure 5 than in Figure 4, because the option to merge weakens the punishments that enforce collusion and thus weakens collusion, it remains highly stable in the merger case. In addition, observe that there are two nodes where the stability flips between the collusion node and the merger node: the unstable one in the entrant and incumbent region of the equilibrium manifold, and the additional stable node in the contestability region nearer the price war or merger node. [22] Comparing the stability

[21] Sannikov (2007) studies stability for the partnership example (see his Figure 2), but not for the duopoly example.

[22] As will be apparent in the simulations of the model, the extra stable node in the contestability region has little

**Figure 4. No-Merger equilibrium manifold and stability**




This figure plots the no-merger equilibrium manifold (blue, solid line). The gray arrows indicate the stability of the dynamic game, where the length of each arrow is the scaled drift of the value state vector. The static Nash equilibrium's output choices of (5,5) are depicted by $N$ in terms of the continuation values of $(50, 50)$. We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

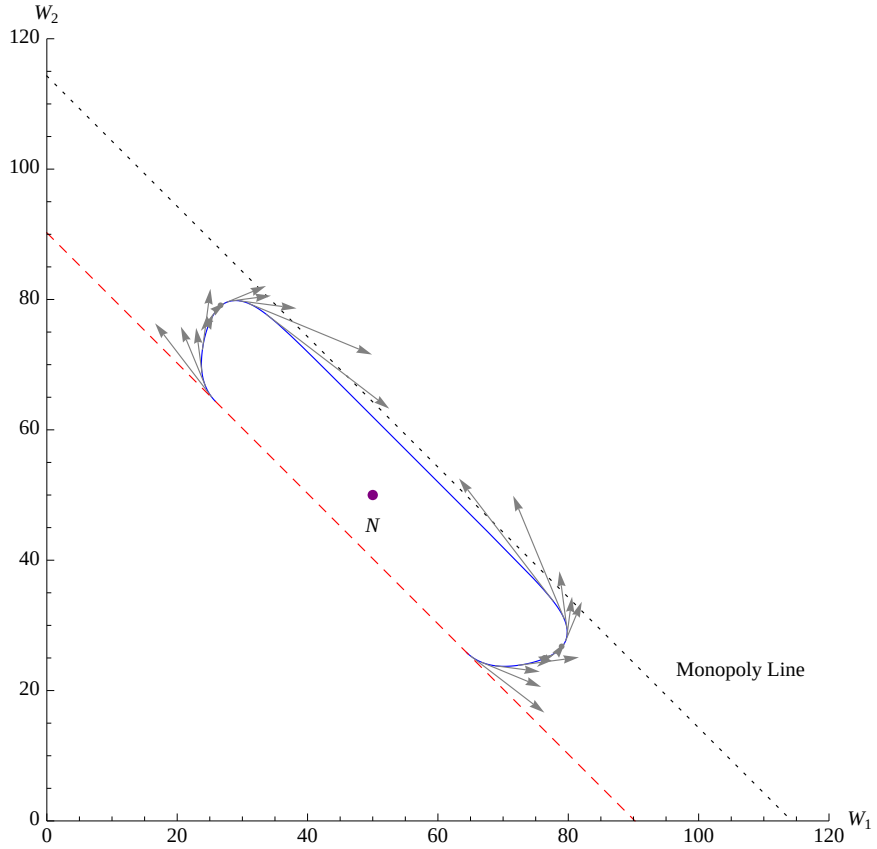
diagrams, in both figures the instability of the contestability region makes it likely that the firms will get back to cooperating if they stray into this region. In the unlikely event that the cusp in the contestability region is crossed, a price war (or a merger if there is the potential for it) is unstable, thus making a price war (or a merger) unlikely.[23]

Importantly, the area close to the merger points in Figure 5 is unstable. As a result, dynamic optimizing behavior of the firms actually delays the onset of mergers (i.e., makes them rare). As the firms approach one of the merger points, they tend to revert back to the collusive region of the equilibrium manifold. This is also illustrated by Figures 6 and 7, which display simulated time-series

impact on the actual dynamics of the equilibrium.

[23]However, if the merger state is approached, then in terms of corporate practice, this corresponds to mergers being "imminent" or "anticipated" just before they are announced. For example, Edmans, Goldstein, and Jiang (2012) and Cornett, Tanyeri, and Tehranian (2011) provide empirical evidence for this anticipation.

**Figure 5. Merger equilibrium manifold and stability**

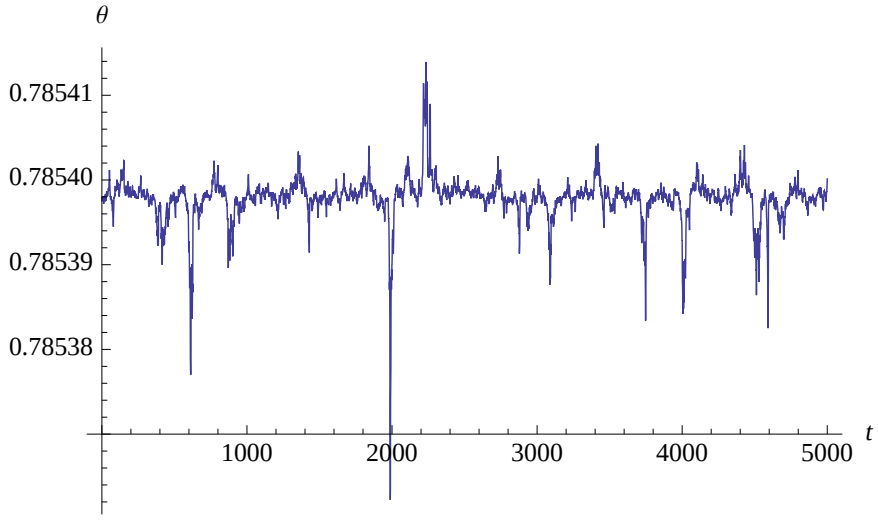



This figure plots the merger equilibrium manifold (blue, solid line), the monopoly line (black, dotted line), and the merger line (red, dashed line) for a merger cost of $k = 24$. The gray arrows indicate the stability of the dynamic game, where the length of each arrow is the scaled drift of the value state vector. The static Nash equilibrium's output choices of (5,5) are depicted by $N$ in terms of the continuation values of $(50, 50)$. We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

samples of play for 5,000 periods in the no-merger and merger cases, respectively. In each figure, the period is on the horizontal axis and the polar coordinate angle, $\theta$, characterizing the region of the equilibrium manifold, is on the vertical axis. Thus, collusion corresponds to $\theta = \frac{\pi}{4} \approx 0.7854$, and the full price war punishment region to $\theta = \frac{5\pi}{4} \approx 3.9270$ or $-\frac{3\pi}{4} \approx -2.3562$. It is apparent that in the no-merger case of Figure 6, divergence from the collusion region into the punishment region is rare, and also brief. In the merger model, mergers occur at the angle corresponding to the punishment regimes, (i.e., at the merger points $\theta = \frac{5\pi}{4}$ or $-\frac{3\pi}{4}$). It is evident from Figure 7 that in case of mergers collusion is also highly stable, with only rare movements towards the merger angles, although the weakening of punishments increases the frequency of these movements slightly.[24]

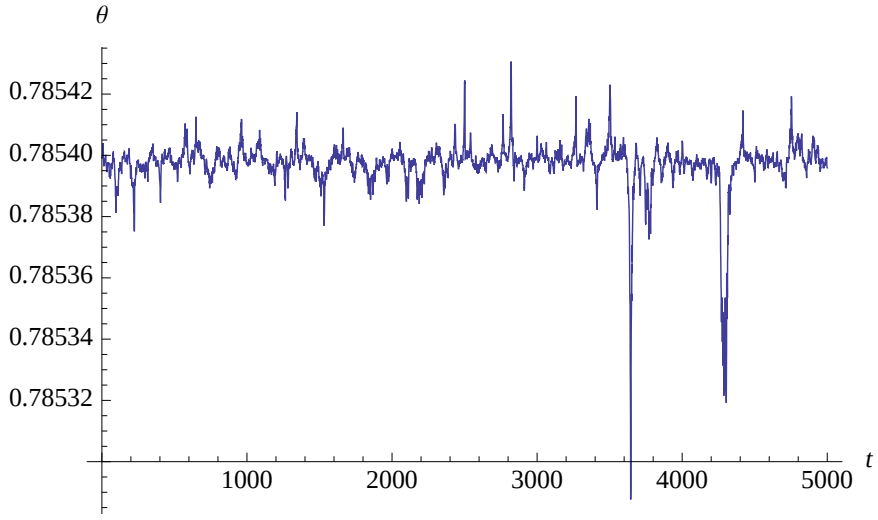
[24] A histogram of occupancy times for all of the regimes, corresponding to ranges of the angles $\theta$, would be so lopsided toward the collusion state that little insight could be drawn from it. Also, to simulate the merger equilibrium

**Figure 6. History of collusion and punishment: No-Merger Case**




This figure plots a simulated history of the non-merger equilibrium for 5,000 draws. The vertical axis indexes $\theta$, the polar-coordinate angle corresponding to points on the equilibrium manifold. Collusion corresponds to $\theta = \frac{\pi}{4}$, and the full price war punishment region to $\theta = \frac{5\pi}{4}$ or $-\frac{3\pi}{4}$. It is evident that excursions away from collusion into the punishment states are extremely rare and brief. We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

**Figure 7. History of collusion and punishment: Merger Case**




This figure plots a simulated history of the merger equilibrium for 5,000 draws. The vertical axis indexes $\theta$, the polar-coordinate angle corresponding to points on the equilibrium manifold. Collusion corresponds to $\theta = \frac{\pi}{4}$, and the merger points are $\theta = \frac{5\pi}{4}$ or $-\frac{3\pi}{4}$. It is evident that excursions away from collusion and toward the merger points are extremely rare. We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

we follow the equilibrium as if firms could de-merge once they have merged; however, a sequence of independent histories, using starting points distributed around the equilibrium manifold and terminating at the merger, would similarly point to the stability of collusion, but even more strongly.

Taken together, the dynamic game's analysis reveals that, as the merger cost falls, the equilibrium manifold changes generally. It flattens, reflecting that the firms are increasingly acting like a shadow monopoly in terms of output, with the main issue being the equity shares in the merged entity. Outsiders unaware of the potential for a merger attempting to value the companies would find output choices diminished relative to the theoretical prediction of the static Nash equilibrium. In addition, regulators would find greater collusion than would seem warranted by that same benchmark. This collusion will be strongest when the merger is most remote. For practical purposes, the merger will be a phantom, seemingly unrelated and hidden from the firms' current actions. While Andrade, Mitchell, and Stafford (2001), for example, point out that stronger antitrust laws and stricter enforcement have provided challenges for anti-competitive mergers, this model's solution implies that the dearth of market power increasing mergers need not imply more competition in a dynamic duopoly game, which is designed for the companies to compete.

Existing merger theory largely builds on static considerations in the sense that, at each instance, both theory and practice would contemplate a merger the moment it occurs. In contrast, merger incentives arise endogenously in our model dynamically when firms sufficiently deviate from their collusive actions, which in turn influences pre-merger strategies.[25] Consequently, the impact of mergers on rival firms' profits is nil or negative in relation to changes in industry concentration, as found by Eckbo (1985), who doubts that the firms were already acting monopolistically prior to the merger. Similarly, Eckbo (1983) states that his results are inconsistent with the market power doctrine and consistent with the efficiency argument: "Thus, if mergers typically take place to realize efficiency gains, we cannot conclude that the 'synergy' effect is expected to produce a significant expansion of the merging firm's share of the market along with an increase in industry rate of output. If scale economies are involved, then these seem on average to be insufficient to make the rivals worse off. Furthermore, the same evidence contradicts the argument that the merging firms were expected to initiate a (monopolistic) 'predatory' price war after consummation of the merger." (p. 272)

From a methodological standpoint, our dynamic model casts doubt on the tests that lead to rejection of the market power doctrine. In fact, there is not much to deter if the anti-competitive effects of horizontal mergers are anticipated in merging and rival firms' product market strategies prior to merger announcements (or likely challenges by regulators). Consistent with our dynamic model's insight, Eckbo (1992) even concludes the following on p. 1005:

[25] Recently, Ahern and Sosyura (2011) provide support for a dynamic view by examining pre-merger strategic manipulation of news releases that affect the firm's stock prices and hence the outcome of merger negotiations.

> While the U.S. has pursued a vigorous antitrust policy towards horizontal mergers over the past four decades, mergers in Canada have until recently been permitted to take place in a virtually unrestricted antitrust environment. The absence of an antitrust overhang in Canada presents an interesting opportunity to test the conjecture that the rigid market share and concentration criteria of the U.S. policy effectively deters a significant number of potentially collusive mergers. The effective deterrence hypothesis implies that the probability of a horizontal merger being anti-competitive is higher in Canada than in the U.S. However, parameters in cross-sectional regressions reject the market power hypothesis on samples of both U.S. and Canadian mergers. Judging from the Canadian evidence, there simply isn't much to deter.

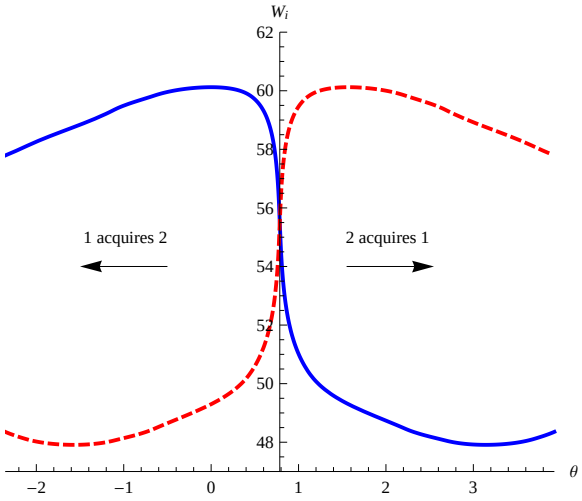
In sum, the key insights from the model analysis help us better understand several important regularities in the mergers and acquisitions literature that have heretofore led to rejection of the market power doctrine. The solution suggests an alternative interpretation of the literature's empirical tests. According to our dynamic model with the possibility of anti-competitive mergers, it is not surprising but rather inevitable that the evidence for the market power doctrine is weak when using capital market data and short-term announcement return methods to gauge changes in competition (or concentration) that have already taken place prior to the announcement return window when firms optimize dynamically.

### 5.4 Pre-merger stock returns

Before considering pre-merger stock returns, it is helpful to consider stock prices throughout the game as a function of the computational state variable that defines the location of the two firms on the equilibrium manifold. Using again the baseline parameter values, Figure 8 plots the pre-merger continuation values, or stock prices, of firm 1 (blue, solid line) and firm 2 (red, dashed line) against the polar coordinate angle $\theta$. The merger cost increases from the top-left panel ($k = 24$) to the top-right panel ($k = 34$), and further from the bottom-left panel ($k = 44$) to the bottom-right panel ($k = 67.2$). There are several interesting patterns for the firms' pre-merger stock prices. First, the payoffs at the endpoints, where the firms are merging (i.e., at the left and right margins of the plots), are generally asymmetric across the firms. Only in the lower right panel, where the merger cost is highest, we can see that the payoffs (and hence shares in the emerged entity) are not asymmetric (see discussion of merger shares in Section 4.2). We will therefore refer to the less valuable firm as the target and the more valuable firm as the acquirer when we analyze merger returns. Second, the firms'

stock prices are equal in the middle of the market-sharing regime at the angle $\theta = \frac{\pi}{4} \approx 0.7854$, where the vertical axis crosses the horizontal axis. It is also striking that the stock price of the acquiring firm first rises, as expected, but then gradually declines as it gets sufficiently close to the merger boundary. Finally, the range of continuation values increases as the fixed cost of merging rises.

**Figure 8. Pre-merger stock prices**



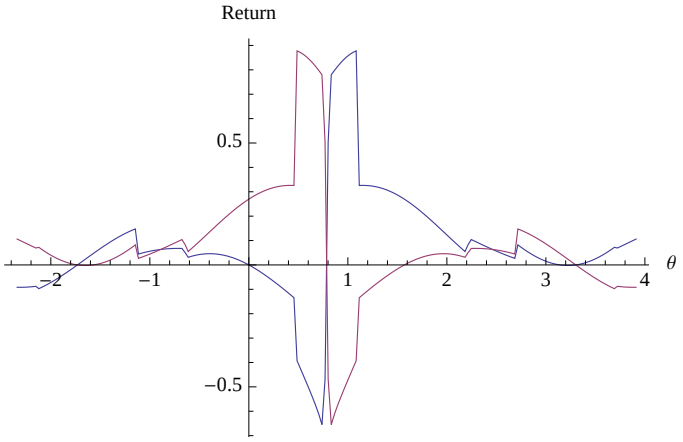

This figure plots the pre-merger continuation values (or stock prices), $W_i$, of firm 1 (blue, solid line) and firm 2 (red, dashed line). Firm 1 acquires 2 when moving clockwise (on the left) but firm 2 acquires 1 when moving counterclockwise (on the right) away from the center of the merger equilibrium manifold of most collusive outputs (i.e., from $\theta = 0.79$ or 45 degrees). The merger cost increases from the top-left panel ($k = 24$) to the bottom-right panel ($k = 67.2$). We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

Recalling that the bottom-right panel corresponds to the no-merger equilibrium from Figure 1, the analysis indicates that there are notable differences between firms' price patterns in the merger and in the no-merger equilibrium. For example, the acquisition price can be higher than the stock prices most closely observed before the announcement of the deal. This is just another way of saying that there is locally a positive pre-merger return for targets in our model, which leads us to our next implication regarding pre-merger returns.

To study pre-merger return dynamics, we approximate instantaneous returns with the local percentage change in value at each point on the equilibrium manifold. Figure 9 plots these quantities against the polar coordinate angle $\theta$. The inner range of $\theta$ represents the normal operation of the firms: locally, one firm gains while the other loses as they approach the center of the cooperation region, which lies on the 45 degree line in the previous figures. At the outer extremes, the merger is imminent. It is apparent that the larger firm, which we loosely label the acquiring firm, exhibits

locally negative pre-merger returns, and the acquired firm has locally positive pre-merger returns. In addition, the figure reveals that the acquiring (acquired) firm first experiences an extended run-up (run-down) as both firms move away from the center of the merger equilibrium manifold of the most collusive outputs (i.e., from the angle $\theta = \frac{\pi}{4} \approx 0.7854$ or the 45 degree angle in Figure 5).

**Figure 9. Pre-merger stock returns**




This figure plots the instantaneous pre-merger stock returns, $dW_i/W_i$, to firm 1 (blue, solid line) and firm 2 (red, dashed line). Firm 1 acquires 2 when moving clockwise (on the left) but firm 2 acquires 1 when moving counterclockwise (on the right) away from the center of the merger equilibrium manifold of collusive outputs (i.e., from $\theta = 0.79$ or 45 degrees). The merger cost increases from the top-left panel ($k = 24$) to the bottom-right panel ($k = 67.2$). We use the baseline environment in which $\Pi_1 = 30$, $\Pi_2 = 30$, $\beta_1 = 2$, $\beta_2 = 2$, $\delta_1 = 2$, $\delta_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 1$, and $r = 1$.

Consistent with these findings, Schwert (2000) and Andrade, Mitchell, and Stafford (2001) document that the average [–20, Close] return to target firms is positive but the average [–20, Close] return to acquiring firms is negative over the 1973–98 period. Note that because we do not model idiosyncratic noise, by assumption our returns do not reflect any systematic risk factors. Moreover, we do not model any other incentives for merging, such as economies of scale, removal of entrenched management, behavioral biases to overinvest by making acquisitions, co-insurance and diversification, etc. The economic magnitude of our pre-merger returns is therefore expected to be moderate.[26] While the stock returns to targets tend to be higher than predicted by our theory, the negative stock returns to acquirers follow more closely the ones observed in practice but are uniquely produced by a dynamic duopoly with imperfect information.[27]

[26] Bradley and Sundaram (2006) also study merger activity in the 1990s. One of their findings is that the importance of "agency costs" or "hubris theory" is perhaps overstated because they apply only to a small minority of cases.

[27] The results reported in Schwert (2000) and Andrade, Mitchell, and Stafford (2001) are consistent with those reported in other studies, such as Jensen and Ruback (1983), Jarrell, Brickley, and Netter (1988), and Moeller, Schlingemann, and Stulz (2005).

# 6 Conclusion

We study mergers in a dynamic noisy collusion model (Green and Porter, 1984; Abreu, Pearce, and Stacchetti, 1986; Sannikov, 2007). At each instant, firms either privately choose output levels or merge, which trades off benefits of avoiding price wars against the costs of merging. Mergers are optimal when collusion fails (i.e., firms sufficiently deviate from a collusive regime). Moreover, long periods of collusion are likely, because colluding is dynamically stable. Therefore, mergers are rare. Finally, mergers (and, in particular, lower merger costs) decrease pre-merger collusion, as punishments by price wars are weakened. Thus, although anti-competitive mergers harm competition ex-post, barriers and costs of merging due to regulation should be reduced to promote competition ex-ante.

We close by noting areas that warrant future research in this class of dynamic models. First, we have restricted attention to two firms. Extending our analysis to three or more firms would be informative about the impact of mergers on non-merging rivals as examined in many of the empirical studies. Moreover, Salant, Switzer, and Reynolds (1983), for example, show that the presence of three or more firms in an industry can deter mergers, and that mergers can be welfare-enhancing, even in the absence of scale economies or synergies. Perry and Porter (1985) examine this result further with a more fine-grained treatment of the allocation of costs in the merged firm and moved the conclusion back in the classical direction. Farrell and Shapiro (1990) establish that quantity competition in a post-merger industry raises prices if there are no scale economies or synergies, but still find cases where mergers are deleterious to potential merging firms. Other researchers, such as Deneckere and Davidson (1985) and Gaudet and Salant (1992), analyze welfare and policy implications in extensions of these models, again finding some counterintuitive results. The technical challenge in expanding the model to multiple firms is significant, however, in that equilibrium manifolds would reside in higher-dimensional spaces, with a concomitant increase in the computational difficulty of numerical solutions, such as inverting the volatility matrix.

Second, as we show, the model is related to agency. While we (as do many studies) treat firms as black boxes that are able to hide information, one might reinterpret this as more like standard agency construct in which managers hide information from rival firms. This might be expanded to incorporate explicit agency elements into the continuous-time game. With agency explicit, a merger might not eliminate all information asymmetries: we could ask whether the increase in market power effected by the merger is strengthened or weakened, and how pre-merger collusion is affected.

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## Appendix A Derivation of the value process in the first stage

Here is the derivation of Lemma 1.

**Proof:** We establish the result for firm 1. We first apply Ito's lemma to $W^1(X_t^1, X_t^2)$ in generate the stochastic continuation value process of the state:

$$dW^1 \;=\; \left(A^1 W^1_{X^1} + A^2 W^1_{X^2} + \tfrac{1}{2}\sigma_1^2 W^1_{X^1X^1} + \tfrac{1}{2}\sigma_2^2 W^1_{X^2X^2}\right) dt \;+\; \sigma_1 W^1_{X^1} dZ_t^1 \;+\; \sigma_2 W^1_{X^2} dZ_t^2 \,. \quad \text{(A.1)}$$

Notice the resemblance of the terms in the drift to the stage-game payoffs in the Bellman equation. Substituting equation (22) into (A.1) yields a simpler expression for the continuation value process:

$$dW^1 = \left(r\, W^1(X^1, X^2) - r\, g_1(A^1, A^2)\right) dt + \sigma_1 W^1_{X^1} dZ_t^1 + \sigma_2 W^1_{X^2} dZ_t^2 \,. \quad \text{(A.2)}$$

We further modify this equation by using the optimality condition (23) to eliminate the $W^1_{X^1}$ term, replacing $W^1_{X^1}$ with $-r g_{1A^1}$ (i.e., the envelope condition):

$$dW^1 \;=\; \left(r\, W^1(X^1, X^2) - r\, g_1(A^1, A^2)\right) dt - \sigma_1 r g_{1A^1} dZ_t^1 + \sigma_2 W^1_{X^2} dZ_t^2 \,. \quad \text{(A.3)}$$

Dropping the arguments, we find that $W^1$ evolves according to:

$$dW^1 \;=\; r\left(W^1 - g_1\right) dt \;-\; \sigma_1 r g_{1A^1} dZ_t^1 \;+\; \sigma_2 W^1_{X^2} dZ_t^2 \,. \quad \text{(A.4)}$$

This eliminates the explicit influence of the state variable $X_t^1$ from the equation. ■

## Appendix B Converting to polar coordinates

We express the differential equation for the equilibrium manifold in polar coordinates to facilitate computation of numerical solutions. Repeating the introduction of polar coordinate notation from the main text, the normal and tangent to the equilibrium manifold are:

$$\mathbf{N}(\theta) = (\cos(\theta), \sin(\theta)) \,, \qquad \mathbf{T}(\theta) = (-\sin(\theta), \cos(\theta)) \,. \quad \text{(B.5)}$$

We can express the first-order cross-partial of $W^1$ in trigonometric form:

$$W^2_{W^1} = \frac{dW^2}{dW^1} = -\frac{\cos(\theta)}{\sin(\theta)} \,. \quad \text{(B.6)}$$

Hence the second-stage Bellman equation (39) becomes:

$$\max_{A^1} \Big\{ r\left(g_1 - W^1\right) - r(g_2 - W^2)\left(-\frac{\sin(\theta)}{\cos(\theta)}\right) + \tfrac{1}{2}\left(-r\frac{\cos(\theta)}{\sin(\theta)} g_{1A^1}\right)^2 W^1_{W^2W^2} + \tfrac{1}{2}\left(r g_{2A^2}\right)^2 W^1_{W^2W^2} \Big\} \,, \quad \text{(B.7)}$$

which then leads to:

$$W^1_{W^2W^2} = \max_{A^1} \left\{ \frac{\left(g_1 - W^1\right) - \left(g_2 - W^2\right)\left(-\frac{\sin(\theta)}{\cos(\theta)}\right)}{r\left(\left(-\frac{\cos(\theta)}{\sin(\theta)} g_{1A^1}\right)^2 + \left(g_{2A^2}\right)^2\right)} \right\} . \tag{B.8}$$

After some algebra, the equation can be restated as follows:

$$W^1_{W^2W^2} = \max_{A^1} \left\{ \frac{\frac{1}{\cos(\theta)}\left(\cos(\theta)(g_1 - W^1) - \left(g_2 - W^2\right)(-\sin(\theta))\right)}{r\cos(\theta)^2\left(\left(\frac{g_{1A^1}}{\sin(\theta)}\right)^2 + \left(\frac{g_{2A^2}}{\cos(\theta)}\right)^2\right)} \right\} , \tag{B.9}$$

or

$$W^1_{W^2W^2} = \max_{A^1} \left\{ \frac{1}{\cos(\theta)^3} \frac{\cos(\theta)(g_1 - W^1) + \sin(\theta)(g_2 - W^2)}{r\left(\left(\frac{g_{1A^1}}{\sin(\theta)}\right)^2 + \left(\frac{g_{2A^2}}{\cos(\theta)}\right)^2\right)} \right\} . \tag{B.10}$$

The numerator term is $\mathbf{N}(g - W)$, and the denominator term is $r|\phi|^2$, just as in Sannikov's formula. Notice that this equation has a curvature on the left-hand side. The fact that it is a curvature will later be used in the numerical solution of the model. We repeat the exercise with firm 2 and obtain:

$$W^2_{W^1W^1} = \max_{A^2} \left\{ \frac{1}{\sin(\theta)^3} \frac{\cos(\theta)(g_1 - W^1) + \sin(\theta)(g_2 - W^2)}{r\left(\left(\frac{g_{1A^1}}{\sin(\theta)}\right)^2 + \left(\frac{g_{2A^2}}{\cos(\theta)}\right)^2\right)} \right\} . \tag{B.11}$$

Notice that the denominators in equations (B.10) and (B.11) are the same.

As shown in Appendix D, the second-order partial derivatives of the continuation values are weighted expressions of the curvature of the equilibrium manifold in the direction of the normal vector, $\kappa(W)$:

$$\cos(\theta)^3 W^1_{W^2W^2} \;=\; \sin(\theta)^3 W^2_{W^1W^1} \;\equiv\; \tfrac{1}{2}\,\kappa(W)\,. \tag{B.12}$$

We can add the two curvature values in equations (B.10) and (B.11) and denote $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ to obtain an expression for the curvature:

$$\kappa(W) \;=\; \max_{A \in \mathcal{A} \setminus \mathcal{A}^N} \frac{2\,\mathbf{N}\,(g - W)}{r|\phi|^2}\,, \tag{B.13}$$

which is Sannikov's (2007) *optimality equation*. Note that the maximization in equation (B.13) is over both $A^1_t$ and $A^2_t$ (excluding the set of pure strategy Nash equilibria, $\mathcal{A}^N$, as mentioned earlier). This is innocuous here because the numerator and denominator are each additively separable in $A^1_t$ and $A^2_t$, so separate maximization for each firm taking its turn as the "principal" is satisfied. ■

# Appendix C Why maximizing the Bellman equation is equivalent to maximizing a ratio in the curvature ODE

Having established that the "agency" optimization problem in equation (39) is equivalent to the ODE in equation (B.10) for the curvature of the equilibrium set boundary $\partial\mathcal{E}$, we need to show why the optimization of the Bellman equation is equivalent to the optimization of the ratio in the ODE.

Consider the abstract problem:

$$\max_x \{f(x) + A\, g(x)\} \,. \tag{C.14}$$

The first-order condition is:

$$f'(x) + A\, g'(x) = 0\,, \tag{C.15}$$

or:

$$A = -\frac{f'}{g'}\,. \tag{C.16}$$

Now consider the maximization problem:

$$\max_x \frac{f(x)}{g(x)}\,. \tag{C.17}$$

The first-order condition can be written as:

$$\frac{f}{g} = \frac{f'}{g'}\,, \tag{C.18}$$

and therefore, at the maximum, we have that:

$$\max_x \frac{f}{g} = \frac{f'}{g'}\,. \tag{C.19}$$

Therefore,

$$A = -\frac{f'}{g'} = -\max_x \frac{f}{g}\,. \tag{C.20}$$

Thus, the maximization of the ratio generates the same optimum (adjusted for the sign) as the Bellman equation. ■

## Appendix D Curvature equality

We want to show that:

$$\cos(\theta)^3\, W^1_{W^2W^2} \;=\; \sin(\theta)^3\, W^2_{W^1W^1}\,. \tag{D.1}$$

To begin, note that:

$$\frac{d}{dW^1}W^1_{W^2} = W^1_{W^2W^2}\frac{dW^2}{dW^1} \;=\; -\,W^1_{W^2W^2}\frac{\cos(\theta)}{\sin(\theta)}\,. \tag{D.2}$$

This is equal to:

$$\frac{d}{dW^1}\frac{1}{W^2_{W^1}} \;=\; -\,\frac{1}{\left(W^2_{W^1}\right)^2}W^2_{W^1W^1} \;=\; -\,W^2_{W^1W^1}\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2\,. \tag{D.3}$$

Equating the two terms and performing algebra yields the result. ■

## Appendix E Optimality of the smooth-pasting condition

In this section we prove Lemma 2 and Proposition 4, which establish optimality of the smooth-pasting condition. Our strategy differs a bit from the more well-known approaches such as Dixit (1993); we will show that if the smooth-pasting condition is satisfied then the second-order condition associated with the Bellman equation is locally satisfied at the boundary point characterised by the the smooth-pasting condition. That is, we provide a verification of the sufficiency of local optimality at the merger point. More concretely, the steps in this demonstration are as follows:

(i) Calculate the second-order condition for the Bellman equation from the optimality condition;

(ii) Evaluate the "ratio" version of the Bellman equation at the value-matching point, using the value-matching condition and also the smooth-pasting condition, resulting in a reduced-form expression for the second partial derivative $W^1_{W^2W^2}$;

(iii) Substitute this reduced-form expression for $W^1_{W^2W^2}$ into the second-order condition, as well as the value-matching condition, establishing that the second-order condition is negative.

**Proof:** (Of Lemma 2) Commencing step (i), the reprise of the Bellman equation is

$$\max_{A^1}\left\{r(g_1 - W^1) - r(g_2 - W^2)W^1_{W^2} + \tfrac{1}{2}\left(-rW^2_{W^1}g_{1A^1}\right)^2 W^1_{W^2W^2} + \tfrac{1}{2}\left(rg_{2A^2}\right)^2 W^1_{W^2W^2}\right\}. \tag{E.4}$$

with optimality condition

$$rg_{1A^1} - rg_{2A^1}W^1_{W^2} + \left(\left(-r^2W^2_{W^1}g_{1A^1}\right)g_{1A^1A^1} \;+ \left(r^2g_{2A^2}\right)g_{2A^2A^1}\right)W^1_{W^2W^2} = 0 \tag{E.5}$$

The second-order condition can make use of the quadratic structure of $g_1$ and $g_2$: the third derivatives are zero, so we have

$$rg_{1A^1A^1} - rg_{2A^1A^1}W^1_{W^2} + r^2\left(-W^2_{W^1}(g_{1A^1A^1})^2 \ + (g_{2A^2A^1})^2\right)W^1_{W^2W^2}$$

Now we can substitute the smooth-pasting condition: $W^1_{W^2} = -1$:

$$rg_{1A^1A^1} + rg_{2A^1A^1} + r^2\left(-W^2_{W^1}(g_{1A^1A^1})^2 \ + (g_{2A^2A^1})^2\right)W^1_{W^2W^2}$$

The same holds for the other player: $W^2_{W^1} = -1$:

$$rg_{1A^1A^1} + rg_{2A^1A^1} + r^2\left((g_{1A^1A^1})^2 \ + (g_{2A^2A^1})^2\right)W^1_{W^2W^2}$$

Now recall the structure of the stage game payoff function $g_1$:

$$g_1(a_1, a_2) = a_1(\Pi_1 \ - \ \beta_1 a_1 \ - \ \delta_1 a_2) \ ,$$

and similarly for $g_2$, so that

$$g_{1A^1A^1} = -2\beta_1 - \delta_1 \qquad g_{1A^2A^1} = -\delta_2 \, ,$$

which are both negative by assumption.

Carrying out step (ii), we next transform the system of Bellman equations to ratio form to isolate the second partials. Appendix Appendix C demonstrates by straightforward algebra that the maximization of the following ratio is equivalent to the original maximization in (39):

$$W^1_{W^2W^2} = \max_{A^1}\left\{\frac{(g_1 - W^1) - r(g_2 - W^2)\frac{1}{r}W^1_{W^2}}{r\left(\left(W^2_{W^1}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)}\right\} . \tag{E.6}$$

Substituting $\frac{1}{W^1_{W^2}}$ for $W^2_{W^1}$ yields:

$$W^1_{W^2W^2} = \max_{A^1}\left\{\frac{(g_1 - W^1) - r(g_2 - W^2)\frac{1}{r}W^1_{W^2}}{r\left(\left(\frac{1}{W^1_{W^2}}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)}\right\} , \tag{E.7}$$

which, along with the first-order condition in $A^1$, is an ordinary differential equation (ODE) in $W^1$. Thus, we have converted the Bellman equation from a partial to an ordinary differential equation.

Finally, carry out step (iii), substituting for $W^1_{W^2W^2}$ from the ratio reformulation of the Bellman equation in equation (E.7), evaluated at the value-matching and smooth-pasting point:

$$W^1_{W^2W^2} = \frac{(g_1 - W^1) - r(g_2 - W^2)\frac{1}{r}W^1_{W^2}}{r\left(\left(\frac{1}{W^1_{W^2}}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)} = \frac{(g_1 - W^1) + (g_2 - W^2)}{r\left(\left(\frac{1}{(-1)}g_{1A^1}\right)^2 + (g_{2A^2})^2\right)} = \frac{(g_1 - W^1) + (g_2 - W^2)}{r\left((g_{1A^1})^2 + (g_{2A^2})^2\right)} \tag{E.8}$$

If we can demonstrate that the numerator of this expression is negative or zero then we will have demonstrated that the second-order condition holds at the smooth-pasting point.

**Lemma 4** *At the smooth-pasting point, we have that:*

$$(g_1 - W^1) + (g_2 - W^2) = 0\,.$$

**Proof:** The expressions $(g_1 - W^1)$ and $(g_2 - W^2)$ are the drifts of the continuation value processes for $W^1$ and $W^2$, respectively (see equation (40)). At the smooth-pasting point, these drifts necessarily are equal and of the opposite sign in order to point along the merger line, which has a slope of $-1$, and they therefore sum to zero. (Equivalently, the curvature at the smooth-pasting point, $W^1_{W^2W^2}$, is zero.) ■

This completes step (iii), establishing the result. ■

We can use this result to prove global optimality.

**Proof:** (of Proposition 4) The optimality of the main action $A^1_t$ as the solution of the HJB equation (39) follows directly from conventional optimal control considerations. Therefore the optimal value state path satisfies equation (B.10); further derivations lead to the differential equation (48), combined with equation(Appendix B), as demonstrated in Appendix B. Sannikov (Sannikov (2007) Theorem 2, p. 1309) establishes that any optimal path must satisfy this differential equation system.

The remaining issue is to provide two boundary conditions for the second-order differential equation, (B.10), that the optimal path necessarily satisfies; the value-matching condition (41) provides one boundary condition. By Lemma 2, the smooth-pasting condition is locally optimal. Suppose that an alternative optimal path exists that does not satisfy the smooth-pasting condition. In that case, there is a kink at the merger boundary, and therefore local optimality cannot be satisfied. Any equilibrium path must satisfy optimality, and therefore any equilibrium path satisfies smooth pasting.

Note that the kink is "one-sided:" the local curvature at the kink point is *positive* infinity, but cannot be negative infinity because the differential equation can approach the merger line from above. Therefore it is only necessary only to observe that the violation of the relevant inequality rests on the positive infinity property. ■

We remark that our results have not established uniqueness of the equilibrium path. We know from numerical experiments that two manifolds that satisfy the smooth-pasting condition exist, with a smaller one fully contained within the larger one that we have analyzed here.

The "standard" proof of the optimality of the smooth-pasting point such as in Dixit (1993) uses a Taylor expansion of the solution of a second-order differential equation, exploiting the structure of the solution stemming from the assumption of fixed coefficients. Our model does not lead to an equation with fixed coefficients, so it is not possible to use Dixit's approach directly. Notice, however, that the smooth-pasting condition is *locally* optimal at the smooth-pasting point; by continuity of the solution that obeys the main differential equation (which would be easy to establish) this optimality argument must also hold in a neighborhood of the smooth-pasting point. (The fixed coefficients assumption would hold locally in a neighbourhood of the fixed point and could thus be applied locally and then possibly expanded globally using continuity.)

It is important to note that Dixit's approach uses a local argument but establishes the *global* optimality of the the smooth-pasting condition. Our argument so far is purely local. Indeed, numerical experiments establish that there are multiple solutions for self-generating manifolds that satisfy the smooth-pasting condition. ■

## Appendix F A remark on social welfare

The continuation values in the model are discounted profits, so we can infer that when these profits increase, the firms are colluding to a greater extent, and consumer surplus is concomitantly reduced. Thus, social welfare is inversely related to the firms' joint profits.

The smooth-pasting equilibrium has lower firm profits when the firms are in the collusion phase than they do if they can never merge, that is, the equilibrium manifold moves in the southwesterly direction relative to the no-merger manifold.

However, the merger cost itself is part of the deadweight loss.[28] How do we incorporate this dead weight loss in the accounting? The answer is that we can ignore it, because the firms' discounted

[28] We are grateful to Mikhail Panov for raising this question.

profits (their continuation values) in the collusion phase are still close to the monopoly line, despite the discounted merger cost. Thus, this deadweight cost is, from the firms' perspectives, just an alternative cost like the cost they would face if they entered into the price war in the no-merger model. ■

## Appendix G Refusals to merge, punishment and jumps

In this appendix[29] we explore the impact of assumptions about the structure of the underlying game, especially the details of the moment of the merger, on outcomes. We focus on one issue in particular: the consequences of the fact that the decision to merge is observable to the rival firm. We emphasize that our discussion is informal.

One of the key assumptions in the main text is that the firms are committed to merge when the value states attain a point on the merger line. Merging at that point is a binary decision, and each firm can observe whether or not the other firm has merged with it because in the event of the merger they subsequently share monopoly profits.

Because the decision by each firm to merge is observable, if the firms do not commit to the merger in advance, the possibility that a firm would refuse to merge can also be admitted as a strategy, and the continuation of the game must be specified in that case. Because the refusal to merge is observable, the structure of the game at that moment and the associated continuation are very different from the noisy collusion game that has been played up to that point.

If one firm refuses to merge, then, given the noncooperative environment of the game, it is appropriate to consider how the other firm would subsequently punish it. One can broadly describe this punishment: it would be to revert to an equilibrium with the worst possible outcome for the firm that has refused to merge.

Notably, the punishment phase must be an equilibrium. Given that the continuation game would not differ from the collusion game in the key respect of having actions that are obscured by noise, the punishment phase would be constituted from the same elements as the pre-merger (or pre-refusal) game. If such an equilibrium exist, we refer to it as a refusal-punishment (R-P) equilibrium.

What we will establish here is that credible punishment phases cannot in fact themselves be equilibria, and so alternative equilibria that are sustained by such punishments cannot exist.

[29]The ideas and contributions of Mikhail Panov, Yuliy Sannikov, and Andrzej Skrzypacz led to the discussion in this appendix.

**The refusal-punishment construction**

For a refusal-punishment equilibrium to function, there must be agreement between the firms about the punishments in the punishment game. These punishments, and in essence the structure of the continuation game, must be agreed to in a pre-play phase of the game the firms play in advance of the noisy collusion game. Thus, there are really three phases of the overall game: (i) the pre-play phase in which the punishments that will be coordinated upon in the event of a refusal to merge; (ii) the pre-merger phase of Sannikov's noisy duopoly game; (iii) the punishment phase in the event that a firm refuses to merge, which itself must be an equilibrium of a pre-merger Sannikov game, including the potential for repeated refusal-punishment phases. Thus, in this final stage, any boundary conditions entailed by the potential for future mergers, or the lack thereof, must be delineated.

In the sequel we first examine the simplest case: that if a merger has been refused and punishment invoked, there is no further potential to merge: the punishment is permanent.

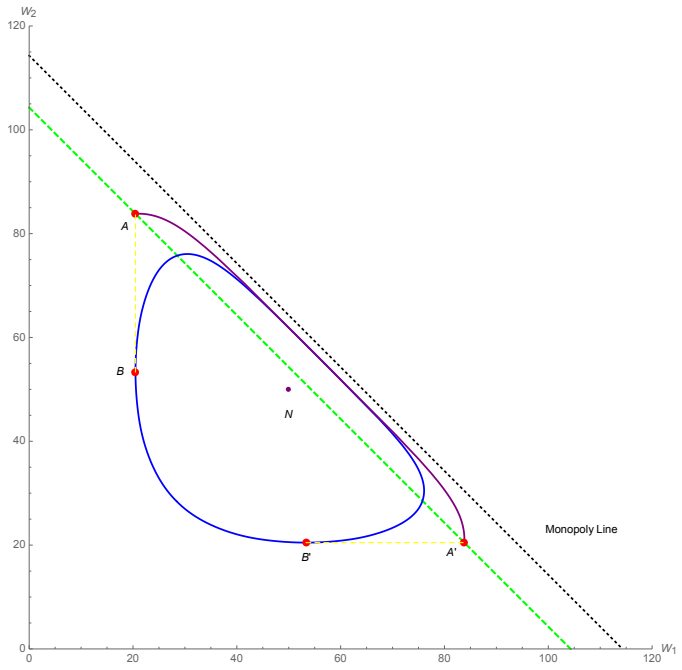
If there is no potential to merge, then the game reverts to the collusion game described by Sannikov (2007): the value states of the firms stochastically and continuously transit around an egg-shaped manifold with payoffs that are strictly bounded by the monopoly line. The initial point on the manifold is determined by the extreme punishment: if firm 1 (with value state on the horizontal axis) has refused to merge, the worst continuation on the no-merger manifold is the leftmost point on it (point $B$ in the figure); similarly if firm 2 (with value state on the vertical axis) has refused to merge, the continuation commences on the lowest point on the manifold (point $B'$ in the figure).

Punishments alter the boundary conditions of the pre-merger game. At the merger point, the punishment must be weakly worse than the worst state at the moment of the merger. Because the no-merger manifold is fixed in size and location, the leftmost and bottom-most punishment points on the manifold are fixed. This in turn dictates the corresponding upper and lower merger points on the merger line (points $A$ and $A'$ in Figure 10). Finally, because the equilibrium pre-merger manifold must be continuous and obey the underlying main differential equation, the pre-merger manifold must land on these two points, and is thus fully and uniquely determined.[30]

Because the landing points are determined by potential punishments, the smooth-pasting condition no longer determines the boundary conditions. The resulting manifold is semicircular and is not tangent at the landing points. We will refer to this manifold as the refusal-punishment

[30] Of course in equilibrium the punishments are not carried out. If they were carried out, the equilibrium manifold would then have to terminate on the punishment point, and this would in turn eliminate the original manifold from consideration.

**Figure 10. The refusal-punishment equilibrium**




In this figure one of the firms can refuse to merge; thereafter it is assumed that no merger is possible. If a firm refuses to merge, say at point $A$, then a punishment is initiated via a jump to point $B$, with symmetric possibilities at points $A'$ and $B'$. The punishment manifold is simply the original no-merger manifold, with play evolving along this manifold as in the original Sannikov no-merger model.

(R-P) manifold. In addition to not being tangent at the merger line, the R-P manifold lies entirely between the no-merger manifold and the monopoly line. This is in contrast to the smooth-pasting manifold, which lies entirely inside the no-merger manifold.

There are *two* ingredients in this construction. The first is that the refusal and punishment are *observable*, unlike the pre-merger play, and also unlike the no-merger manifold that the firms jump to if the punishment is carried out.

The second is that the punishment entails a *jump*. This is key: in the other stages of the game, jumps are not possible because actions are confounded by noise. But the *observability* of the refusal means that the game is no longer in its noisy mode at that instant, and a more conventional, static, full-information game can be played. Because there is a jump if the punishment is invoked, the punishment is far more severe than the incremental punishments of the pre-merger game. As the cost shrinks, the size of the jump interval actually increases, making the jump punishments even stronger. This is why the R-P manifold achieves better cooperation than the no-merger game alone can achieve, if it is an equilibrium.

**The punishments are not renegotiation-proof**

The above reasoning breaks down for a clear reason: the punishment phase of the equilibrium would entail the firms traversing the punishment manifold that lies *below* the merger line. Every point on the punishment manifold has the property that it is Pareto-dominated by some point on the merger line. One of the firms could propose to move to such a point from the punishment manifold by merging, and the other firm could accept this proposal and improve its payoff. This logic works because the proposal to merge, unlike the production action, is publicly observable. Therefore the punishment manifold is not renegotiation proof, so in turn the punishment cannot be an equilibrium. This leaves only the manifolds that satisfy the smooth-pasting property as potential equilibria: because the smooth-pasting manifolds lie *above* the merger line, it is not Pareto-improving to immediately jump to the merger line. Thus, the R-P manifolds are ruled out. ■

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